Topics in Combinatorics 2011

Homework 3 (due October 28)

1. Find the exponential generating function of the number of permutations of n letters that have an odd number of cycles, each of which is of even length.

2. Fix the integer k. Let $f(n,k)$ be the number of permutations of n letters whose cycle lengths are all divisible by k. Find a simple, explicit expression for the exponential generating function for $(f(n,k))_{n\geq 0}$. Find a simple explicit formula for $f(n,k)$.

3. Find the exponential generating function for the number of labeled bipartite graphs.
   Hint: First count 2-colored bipartite graphs, *i.e.* labeled bipartite graphs with a coloring of the vertices in two colors with the property that connected vertices are colored with different colors.

4. In this exercise, we prove Cayley's theorem: t_n, the number of labeled trees on n vertices, equals n^{n-2}. Show that the left-hand side of

   $$n^2 t_n = n^n$$

   counts directed paths (possibly of zero length) of labeled vertices with some trees (possibly trivial ones with one vertex) hanging from them. Show that the right-hand side counts collections of directed cycles (possibly of zero length) of labeled vertices with some trees (possibly trivial ones with one vertex) hanging from them. Find a bijection between both types of objects.
   Note: A more famous bijective proof of Cayley's theorem is via *Prüfer codes*.