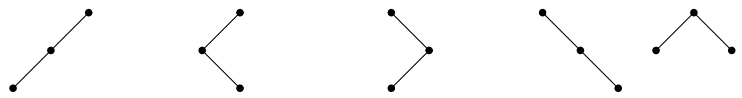
Topics in Combinatorics 2011

Homework 4 (due November 4)

1. A *binary tree* is an unlabeled tree in which every vertex is either a leaf, or it has a left neighbor, or a right neighbor, or both a left and a right neighbor. The following are all binary trees of size 3.



Let b_n denote the number of binary trees on n vertices. Show that the ordinary generating function $B(x)$ for b_n satisfies the equation

$$B(x) = 1 + xB^2(x).$$

Use Lagrange inversion to find b_n.

2. Expand

$$\prod_{i \geq 1} (1 + x_i + x_i^2)$$

in terms of the elementary symmetric functions e_λ.
Hint. Note that $1 + x + x^2 = (1 - \omega x)(1 - \omega^2 x)$, where ω is the primitive third root of unity.

3. For a partition $\lambda = \langle 1^{m_1} 2^{m_2} \cdots \rangle$, define

$$z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots$$

If a permutation π has a_i cycles of length i, call the partition $\langle 1^{a_1}, 2^{a_2}, \ldots \rangle$ the *type* of π and denote it $\rho(\pi)$.

   (a) Prove that the number of permutations of type λ is $n!/z_\lambda$.
   Hint. Define a map $\varphi \colon \mathfrak{S}_n \to \{\pi \in \mathfrak{S}_n \colon \rho(\pi) = \lambda\}$ that satisfies $|\varphi^{-1}(\pi)| = z_\lambda$.

   (b) Prove that permutations π, σ are conjugate (*i.e.* $\pi = \tau\sigma\tau^{-1}$ for some τ) if and only if they have the same type.

   (c) We know for every finite group G and for every element $x \in G$, we have $|[x]| \cdot |C(x)| = |G|$, where $[x]$ is the conjugacy class of x and $C(x)$ is the centralizer of x. It follows that for every permutation π,

   $$|C(\pi)| = z_{\rho(\pi)}.$$

   Prove this equality bijectively.

4. Prove that

$$\sum_\lambda z_\lambda^{-1} = \sum_\mu z_\mu^{-1} = \frac{(2n-1)!!}{(2n)!!},$$

where the first (respectively, second) sum is over partitions λ (respectively, μ) of $2n$ with even (respectively, odd) parts.