Topics in Combinatorics 2011

Homework 5 (due November 11)

1. Prove the nine statements in the following table.

basis b_λ	$b_\lambda(1,q,\ldots,q^{n-1},0,\ldots)$	$b_\lambda(1,q,q^2,\ldots)$	$b_\lambda(1,1,\ldots,1,0,\ldots)$
e_λ	$\prod_i q^{\binom{\lambda_i}{2}}\binom{n}{\lambda_i}_q$	$\prod_i \frac{q^{\binom{\lambda_i}{2}}}{(1-q)(1-q^2)\cdots(1-q^{\lambda_i})}$	$\prod_i \binom{n}{\lambda_i}$
h_λ	$\prod_i \binom{n+\lambda_i-1}{\lambda_i}_q$	$\prod_i \frac{1}{(1-q)(1-q^2)\cdots(1-q^{\lambda_i})}$	$\prod_i \binom{n+\lambda_i-1}{\lambda_i}$
p_λ	$\prod_{i=1}^{\ell} \frac{1-q^{n\lambda_i}}{1-q^{\lambda_i}}$	$\prod_{i=1}^{\ell} \frac{1}{1-q^{\lambda_i}}$	n^ℓ

Here λ is a partition of length ℓ, there are n ones in $b_\lambda(1,1,\ldots,1,0,\ldots)$, and

$$\binom{n}{k}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}$$

is the q-binomial coefficient.

2. Let $p_n^{(k)} = \sum_\lambda m_\lambda$, where the sum is over $\lambda \vdash n$ of length k. For example, $p_n^{(1)} = p_n$. Show that

$$\sum_{n,k} p_n^{(k)} t^{n-k} u^k = E(u-t)H(t).$$

Conclude that

$$p_n^{(k)} = \sum_{a+b=n} (-1)^{a-k}\binom{a}{k} e_a h_b.$$