Topics in Combinatorics 2011

Homework 7 (due November 25)

1. Define $\varphi\colon \Lambda \to \mathbb{Q}$ by $\varphi(h_n) = n$. Compute $\varphi(e_n), \varphi(p_n)$ for all n.

2. Define a specialization $\varphi\colon \Lambda \to \mathbb{Q}$ by $\varphi(h_n) = p(n) = |\operatorname{Par}(n)|$. Compute $\varphi(e_n), \varphi(p_n)$ for all n. Use the result for $\varphi(p_n)$ to find a recursive formula for $p(n)$.

3. (Bonus problem.) Let $E(t) = \prod_{n\geq 1} \frac{1+t^n}{1-t^n}$.

   (a) Prove that $h_0 = 1$, $h_n = 2$ if $n \geq 1$ is a square, and $h_n = 0$ otherwise. *Hint:* Use *Jacobi's triple product identity*

   $$\sum_{n\in\mathbb{Z}} t^{n^2} u^n = \prod_{n\geq 1} (1-t^{2n})(1+t^{2n-1}u)(1+t^{2n-1}u^{-1}).$$

   (b) Prove that for $n \geq 1$, $p_n = 2(-1)^{n-1}\sigma'(n)$, where $\sigma'(n)$ is the sum of divisors $d \geq 1$ of n for which n/d is odd.

   (c) Write $N_r(n)$ for the number of integer vectors $(x_1, \ldots, x_r) \in \mathbb{Z}^r$ for which $x_1^2 + \ldots + x_r^2 = n$. Use

   $$n!e_n = \begin{vmatrix} p_1 & 1 & 0 & \ldots & 0 \\ p_2 & p_1 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n-1} & p_{n-2} & \cdot & \ldots & n-1 \\ p_n & p_{n-1} & \cdot & \ldots & p_1 \end{vmatrix},$$

   which is equivalent to one of Newton's formulas, and (a), (b) to prove that

   $$N_r(n) = \frac{(2r)^n}{n!} \begin{vmatrix} \sigma'(1) & 1/2r & 0 & \ldots & 0 \\ \sigma'(2) & \sigma'(1) & 2/2r & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma'(n-1) & \sigma'(n-2) & \cdot & \ldots & (n-1)/2r \\ \sigma'(n) & \sigma'(n-1) & \cdot & \ldots & \sigma'(1) \end{vmatrix}.$$

4. A *Gelfand-Tsetlin pattern* is a triangular array G of non-negative integers

   $$\begin{matrix} a_{11} & & a_{12} & & a_{13} & & \cdots & & a_{1n} \\ & a_{22} & & a_{23} & & \cdots & & a_{2n} & \\ & & a_{33} & & \cdots & & a_{3n} & & \\ & & & \ddots & & \iddots & & & \\ & & & & a_{nn} & & & & \end{matrix}$$

   such that $a_{ij} \leq a_{i+1,j+1} \leq a_{i,j+1}$ whenever all three numbers are defined. Find a natural bijection between Gelfand-Tsetlin patterns with fixed first row α of length n and SSYT of shape α^r (α in reverse order) and largest entry at most n.