Topics in Combinatorics 2011

Homework 8 (due December 2)

1. Express $\sum_{\tau \vdash n} (1-t)^{\ell(\tau)-1} m_\tau$ in terms of Schur functions.

2. (a) Prove that the number of paths of length n that start in $(0,0)$, do not go below the x-axis, and have only steps of the form $(1,1)$ and $(1,-1)$, equals $\binom{n}{\lfloor n/2 \rfloor}$.
   Hint. If a path goes below the x-axis and $(i,-1)$ is the first point on the line $y=-1$, reflect the part of the path on $[i,n]$ with respect to $y=-1$.

   (b) Prove that
   $$\sum_{\lambda} f^\lambda = \binom{n}{\lfloor n/2 \rfloor},$$
   where the sum on the left is over all $\lambda \vdash n$ with $\ell(\lambda) \leq 2$.

3. (a) Prove that
   $$\sum_{\lambda \vdash n} f^\lambda = |\{w \in \mathfrak{S}_n \colon w^2 = \mathrm{id}\}|.$$

   (b) Prove that
   $$\prod_i \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_i x_j} = \sum_\lambda s_\lambda(x).$$