Topics in Combinatorics 2011

Homework 9 (due December 9)

1. (a) Let $w \in \mathfrak{S}_n$, $w^2 = \mathrm{id}$, $w \overset{\mathrm{RSK}}{\longmapsto} (P, P)$. Prove that the number of columns of P of odd length is equal to the number of fixed points of w.
   Hint: Let $\nu(i,j)$, $0 \le i, j \le n$, be the partition in position (i,j) of the growth diagram of w. Prove that the number of odd parts of $\nu(i,i)'$ is equal to the number of fixed points k of w with $k \le i$.

   (b) Verify the identity
   $$\prod_i \frac{1}{1 - qx_i} \prod_{i<j} \frac{1}{1 - x_i x_j} = \sum_\lambda q^{c(\lambda)} s_\lambda(x),$$
   where $c(\lambda)$ denotes the number of odd parts of λ'. What do we get if we plug in $q = 0$ into the equality from (b)?

2. (a) In Homework 8, problem 3, you showed that
   $$\sum_\lambda f^\lambda = \binom{n}{\lfloor n/2 \rfloor},$$
   where the sum on the left is over all $\lambda \vdash n$ with $\ell(\lambda) \le 2$. Expand $s_{\lceil n/2 \rceil} s_{\lfloor n/2 \rfloor}$ and rederive this result.

   (b) In Homework 5, problem 2, we defined $p_n^{(k)} = \sum_\lambda m_\lambda$, where the sum is over $\lambda \vdash n$ of length k, and you showed that
   $$p_n^{(k)} = \sum_{a+b=n} (-1)^{a-k} \binom{a}{k} e_a h_b.$$
   Use this result to express $p_n^{(k)}$ in terms of Schur functions.