1.5 Group actions

- **Group action**: Let G be a group, let Ω be a set, and let $\rho\colon G \to \mathrm{Sym}(\Omega)$ be a homomorphism of groups. Then ρ is called a *group action* of G upon Ω.

- **Shorthand notation**: We can avoid explicitly stating the "name" ρ of the action by writing $G \rightsquigarrow \Omega$ instead of ρ, ω^g instead of $\omega^{\rho(g)}$, and g^Ω instead of the permutation $\rho(g)$. This is particularly useful when it is clear from the context which action of G upon Ω we have in mind.

- **The induced permutation group**: The image $\rho(G)$ of ρ is a permutation group on Ω. We shall refer to this group as *the permutation group induced by the action* ρ. Its shorthand notation is G^Ω.

- **The kernel**: The kernel of the action ρ is denoted by $\mathrm{Ker}(\rho)$ (or $\mathrm{Ker}(G \rightsquigarrow \Omega)$) if we want to avoid naming ρ). It consists of all those elements $g \in G$ that induce a trivial permutation of Ω. Note that $G^\Omega \cong G/\mathrm{Ker}(G \rightsquigarrow \Omega)$.

- **Faithfulness**: If $\mathrm{Ker}(\rho) = 1$, then we say that the action is *faithful*. In this case ρ is an embedding of G into $\mathrm{Sym}(\Omega)$, and thus $G \cong G^\Omega$. After identifying each g with $\rho(g)$, we may view G as a permutation group on Ω.

- **Induced action of subgroups**: If $H \le G$ and $\rho\colon G \to \mathrm{Sym}(\Omega)$ is an action, then the restriction $\rho\,|_H\colon H \to \mathrm{Sym}(\Omega)$ is also an action, which is called the *induced action of a subrgoup*. In this sense, whenever G acts on a set Ω, so does each of its subgroups.

- **Permutation groups as actions**: Conversely, if $G \le \mathrm{Sym}(\Omega)$ is a permutation group, then the identity mapping $\iota\colon G \to \mathrm{Sym}(\Omega)$ is a faithful action of G upon Ω. In this sense we may identify notions of *faithful group actions* and *permutation groups*.

Remark. Let Ω be a finite nonempty set, let G be a group and let

$$\cdot : \Omega \times G \to \Omega, \qquad \cdot : (a, g) \mapsto a \cdot g$$

be a mapping that, for all $a \in \Omega$ and $g, h \in G$, satisfies the following axioms:

A1. $a^1 = a$;

A2. $a^{(gh)} = (a^g)^h$.

For each $g \in G$, let $\rho(g)$ be a mapping from Ω to Ω, which maps $a \in \Omega$ to $a \cdot g \in \Omega$. Then $\rho(g) \in \mathrm{Sym}(\Omega)$, and the mapping

$$\rho\colon G \to \mathrm{Sym}(\Omega), \qquad \rho\colon g \mapsto \rho(g)$$

is a group action of G upon Ω.

Conversely: If $\rho\colon G \to \mathrm{Sym}(\Omega)$ is a group action, then the mapping $(a,g) \mapsto a^{\rho(g)}$ satisfies axioms A1 in A2.

1.6 Stabilisers and orbits of group actions

Throughout this section, let G act upon Ω, let $\omega \in \Omega$, and let $\Delta \subseteq \Omega$.

- **Stabiliser**: The set

  $$G_\omega = \{g \in G : \omega^g = \omega\}$$

  is called the *stabiliser* of ω. Similarly:

  $$G_\Delta = \{g \in G : \Delta^g = \Delta\},$$

  where $\Delta^g = \{\delta^g : \delta \in \Delta\}$, s the *set-wise stabiliser* of Δ. On the other hand,

  $$G_{(\Delta)} = \{g \in G : \delta^g = \delta \text{ for each } \delta \in \Delta\} = \bigcap_{\delta \in \Delta} G_\delta,$$

  is called the *point-wise stabiliser* of Δ.

- **Induced action of the set-wise stabiliser**: There is an obvious action of G_Δ upon Δ. The permutation group, induced by this action, is denoted G_Δ^Δ. The kernel of this action is $G_{(\Delta)}$, implying that $G_{(\Delta)} \lhd G_\Delta$ and $G_\Delta^\Delta \cong G_\Delta / G_{(\Delta)}$.

  EXAMPLE. Let $\Omega = \{1, 2, \ldots, n\}$, let $\Delta = \{1, 2, \ldots, m\}$ for $1 < m < n$ and let $G = \mathrm{Sym}(\Omega)$. Then $G_{(\Delta)} \cong \mathrm{Sym}(\Omega \setminus \Delta)$, $G_\Delta \cong \mathrm{Sym}(\Delta) \times \mathrm{Sym}(\Omega \setminus \Delta)$, and $G_\Delta^\Delta \cong \mathrm{Sym}(\Delta)$. ■

- **Orbit**: The set

  $$\omega^G = \{\omega^g : g \in G\}$$

  is called the *orbit* of ω.

- **Transitivity:** If $\omega^G = \Omega$, then the action is *transitive*;

- **Semiregularity**: If $|G_\omega| = 1$ for every $\omega \in \Omega$, then the action is *semiregular*;
- **Regularity**: The action if *regular* if it is transitive and semiregular.
- **Conjugating the stabiliser**: For any $g \in G$ we have:

$$G_{(\omega^g)} = (G_\omega)^g.$$

- **Orbit space**: If we write $\omega \sim \delta$ whenever $\delta \in \omega^G$, it can be proved that $\sim$ is an equivalence relation on Ω whose equivalence classes are precisely the orbits of the action. The set of all orbits

$$\Omega/G = \{\omega^G : \omega \in \Omega\}$$

is called the *orbit space* of the action of G on Ω, and thus constitutes a partition of Ω.

- **Orbit-stabiliser formula**: For any $\omega \in \Omega$ we have

$$|G_\omega|\,|\omega^G| \;=\; |G|.$$

- **Frattini argument**: Suppose that G, which acts on Ω, contains a subgroup H which acts transitively on Ω. Then for any $\omega \in \Omega$: $G = HG_\omega = G_\omega H$. If H happens to acts regularly on Ω, then each $g \in G$ factorises uniquely into a product hg' for $h \in H$ and $g' \in G_\omega$.

Exercise.

- Let G act upon Ω, $|\Omega| = n$, and let $\Delta \subseteq \Omega$, $|\Delta| = k$. Show that $|G : G_{(\Delta)}| \leq n(n-1)\ldots(n-k+1)$.
- Let G act upon a set Ω, let p be a prime divisor of $|G|$, let P be a Sylow p-subgroup of G, and suppose that $|\Omega| = p^k m$ for some integer m coprime to p. Show that every shortest orbit of P has length at least p^k. (Note that this implies that whenever G is transitive and $|\Omega| = p^k$, then P is also transitive.) Show that, in fact, there is an orbit of P of length precisely p^k.
- Use the Frattini argument to show the following classical result in group theory: Let $H \lhd G$ and let P be a Sylow p-subgroup of H. Then $G = N_G(P)H$. (Hint: Consider the action of G on the set of Sylow p-subgroup of H by conjugation.)

1.7 A few examples of actions arising from group theory

- **Action by right multiplication**: Let G act upon the set G by the rule $h^g = hg$. This action is regular.
- **Action by left multiplication**: Let G act upon the set G by the rule $h^g = g^{-1}h$. This action is also regular. (Note that if g^{-1} is replaced by g, then this is no longer an action. What goes wrong?)
- **Action by conjugation**: Let G act upon itself by the rule $h^g = g^{-1}hg$. This action is called the *action by conjugation*. It is never transitive (unless $G = 1$) since $\{1\}$ is always an orbit. The orbits of this action are called *conjugacy classes* of G. Is this action always faithful? What is the kernel of this action? What is the stabiliser of an element $h \in G$?
- **Action by conjugation on subgroups**: Each group G acts upon the set of its subgroups by conjugation: $H^g = g^{-1}Hg$ for every $H \leq G$ and $g \in G$. Is this action ever transitive? Is it ever faithful? What is the stabiliser of a group H?
- **Action on the set of Sylow subgroups**: Let p be a prime divisor of the order of a group G and let $\mathrm{Syl}_p(G)$ be the set of all Sylow p-subgroups of G. Then G acts upon $\mathrm{Syl}_p(G)$ by conjugation: $P^g = g^{-1}Pg$ for every $P \in \mathrm{Syl}_p(G)$. By the Sylow's theorems, we know that this action is transitive.
- **Action on cosets**: Let $H \leq G$ and let $G/H = \{Hg : g \in G\}$ be the corresponding *cosets space* (that is, the set of *right cosets* of H in G). Then G acts upon G/H in the following way:

  $$(Hx)^g = Hxg \text{ for any } Hx \in G/H \text{ and } g \in G.$$

  This action is called the *action of a group on the cosets of a subgroup*. It is easy to check that this action is transitive, that the stabiliser of the element $H \in G/H$ is H (as a subgroup of G) and that the kernel of this action is $\cap_{g\in G} H^g$. The latter groups is also called the *core* of H in G and denoted by $\mathrm{core}_G(H)$.

EXERCISE.

- Suppose that G contains a subgroup H of index n. Show that H contains a subgroup K, which is normal in G and has index at most $n!$ in G. (Note that this shows that every subgroup of index 2 is normal.)

1.8 Transitive constituents

Let $\rho\colon G \to \mathrm{Sym}(\Omega)$ be a group action and let Δ be an orbit of this action. Then one can define the action

$$\rho_\Delta\colon G \to \mathrm{Sym}(\Delta), \quad \omega^{\rho_\Delta(g)} = \omega^{\rho(g)} \text{ for each } \omega \in \Delta, g \in G.$$

This action is clearly a transitive. The induced permution group $G^\Delta = \mathrm{Im}(\rho_\Delta) \leq \mathrm{Sym}(\Delta)$ is then called a *transitive constituent* of the action ρ.

Since the homomorphism $\pi_\Delta\colon G \to G^\Delta$ is not necessarily an isomorphism, G^Δ does not carry all the information about G. What is more, two actions might have the same transitive constituents, but can be still very different:

Example. The permutation groups

$$G = \langle (0,1,2),(3,4,5) \rangle, \quad H = \langle (0,1,2)(3,4,5) \rangle.$$

have the same transitive constituents, but are not isomorphic. ∎

However, the following still holds: Let $\Delta_1, \Delta_2, \ldots, \Delta_r$ be the orbits of a permutation group G. Then there exists a group monomorphism

$$\iota\colon G \hookrightarrow G^{\Delta_1} \times G^{\Delta_2} \times \ldots \times G^{\Delta_r},$$

for which the projection of the group $\iota(G)$ to each of the component G^{Δ_i} is surjective.

Remark. In other words, G is a *subdirect product* of its transitive constituents.