1.9 Isomorphism and equivalence of actions

Let $\rho\colon G \to \mathrm{Sym}(\Omega)$ and $\vartheta\colon H \to \mathrm{Sym}(\Delta)$ be two actions. If

$$f\colon G \to H \text{ and } \varphi\colon \Omega \to \Delta$$

are a group isomorphism and a bijection, respectively, such that

$$\varphi(\omega^{\rho(g)}) = \varphi(\omega)^{\vartheta(f(g))} \qquad \text{for all } \omega \in \Omega \text{ and } g \in G,$$

then the pair (f, φ) is called an *isomorphism of actions* ρ and ϑ. Two permutation groups $G \leq \mathrm{Sym}(\Omega)$ and $H \leq \mathrm{Sym}(\Delta)$ are called *permutationally isomorphic* whenever the corresponding faithful actions are isomorphic.

If $G = H$ and ρ and θ are isomorphic via (id_G, φ), then the two actions are *equivalent*. Let us now state the characterisation of transitive actions.

THEOREM 1.1 *Let $\theta\colon G \to Sym(\Omega)$ be a transitive action and let $\omega \in \Omega$. Further, let $\rho\colon G \to Sym(G/G_\omega)$ be the action of G upon the cosets of G_ω in G by right multiplication. Then the actions θ and ρ are equivalent.*

PROOF. Define

$$\varphi\colon G/G_\omega \to \Omega, \qquad G_\omega h \mapsto \omega^{\theta(h)}$$

and show that φ is well defined, injective and surjective (and thus bijective). Further, check that

$$\varphi((G_\omega h)^{\rho(g)}) = \varphi(G_\omega hg) = \omega^{\theta(hg)} = (\omega^{\theta(h)})^{\theta(g)} = \varphi(G_\omega h)^{\theta(g)}.$$

By definition of isomorphism of actions, we now see that (id_G, φ) is the required isomorphism. ■

This has the following interesting consequence:

PROPOSITION 1.2 *Let $\rho\colon G \to Sym(\Omega)$ and $\sigma\colon G \to Sym(\Delta)$ be two transitive actions of G. Then the following statements are equivalent:*

(i) *The actions ρ are σ equivalent;*

(ii) *For every $\omega \in \Omega$ there is $\delta \in \Delta$, such that $G_\omega = G_\delta$;*

(iii) *There exist $\omega \in \Omega$ and $\delta \in \Delta$, such that $G_\omega = G_\delta$;*

(iv) *For every $\omega \in \Omega$ and $\delta \in \Delta$ the stabilisers G_ω and G_δ are conjugate.*

PROOF. Suppose (i) holds and let (id, φ), $\varphi\colon \Omega \to \Delta$, be an isomorphism of the two actions. That is,

$$\omega^{\rho(g)} = \varphi(\omega)^{\sigma(g)}.$$

Now show that $G_\omega = G_{\varphi(\omega)}$, which implies (ii).

That (ii) implies (iii) is obvious.

To show that (iii) implies (i), use Theorem 1.1.

The equivalence between (iii) and (iv) follows from the fact that stabilisers of transitive actions are conjugate. ■

Similarly for isomorphism of actions:

PROPOSITION 1.3 *Let G act transitively upon Ω and let H act transitively on Δ. Further, let $\omega \in \Omega$ and $\delta \in \Delta$. Then these two actions are isomorphic if and only if there is a group isomorphism $\alpha\colon G \to H$, such that $(G_\omega)^\alpha = H_\delta$.*

PROOF. In view of Theorem 1.1, we may assume that $\Omega = G/G_\omega$, $\Delta = H/G_\delta$, $\omega = G_\omega \in G/G_\omega$, $\delta = H_\delta \in H/H_\delta$ and that G and H act upon Ω and Δ with right multiplication. Denote these two actions by ρ and σ.

Suppose first that ρ and σ are isomorphic, and let (α, φ) be an isomorphism between them. Using the definition of the isomorphism of actions, it is now easy to show that α maps G_ω to H_δ.

Conversely, if $G_\omega = (H_\delta)^\alpha$ for some isomorphism $\alpha\colon G \to H$, then one can define the mapping $\tilde{\alpha}\colon G/G_\omega \to H/H_\delta$ satisfying $G_\omega x = H_\delta x^\alpha$ for every $x \in G$. It is not difficult to check that $\tilde{\alpha}$ is thus well defined and bijective, and that $(\alpha, \tilde{\alpha})$ is an isomorphism between the two actions. ■

EXERCISE.

- Show that equivalence and isomorphism of actions are equivalence relations.
- Prove the equivalence between (i) and (iv) of Propostion 1.2 directly, without using Theorem 1.1. Similarly, prove Proposition 1.3 without appealing to Theorem 1.1.
- Let H and G be two permutation groups on a set Ω. Show that they are permutationally isomorphic if and only if they are conjugate within $\mathrm{Sym}(\Omega)$.
- Find two actions of the same group G that are isomorphic but not equivalent.
- Find all faithful transitive actions (up to isomorphism/equivalence) of the dihedral group D_n (consider two cases: n even and n odd).