Group Theory 2013/14 – homework 1

(1) Let $D_8 = \langle a, b \mid a^4 = b^2 = 1, ab = ba^{-1}\rangle$be the dihedral group of order 8. Check that the assignment
$$a \mapsto \begin{pmatrix} -7 & 10 \\ -5 & 7 \end{pmatrix},\ b \mapsto \begin{pmatrix} -5 & 6 \\ -4 & 5 \end{pmatrix}$$
determines a presentation of D_8 over $\mathbb{C}$. Is this presentation irreducible?

(2) The quaternion group Q_8 of order 8 can be considered as the subgroup of $\mathrm{GL}_2(\mathbb{C})$ generated by the matrices
$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
Find all irreducible representations of Q_8 over $\mathbb{C}$.

(3) Let G be a cyclic group of order n and k a field.
   (a) Suppose that the characteristic of k does not divide n. Prove that kG can be written as a direct sum of pairwise non-equivalent irreducible representations, whose degrees are the same as the degrees of the irreducible factors of $X^n - 1$ in $k[X]$.
   (b) Suppose that n is prime and $k = \mathbb{Q}$. Show that G has an irreducible representation S of degree $n-1$, and that $\mathrm{End}_{kG}(S) \cong \mathbb{Q}(e^{2\pi i/n})$.

(4) Let $n > 1$ and $x \in A_n$. Prove the following:
   (a) If x commutes with some odd permutation of S_n, then the conjugacy class of x in A_n is the same as the conjugacy class of x in S_n.
   (b) If x does not commute with any odd permutation in S_n, then the conjugacy class of x in S_n is a disjoint union of the conjugacy class of x in A_n and the conjugacy class of $(1\,2)x(1\,2)$ in A_n.

This homework is due 11 November 2013.