Group Theory 2013/14 – homework 5

(1) Let G act transitively on the set Ω. Let π be the corresponding permutation character. Assume that $\pi(g) \leq 1$ for every $g \neq 1$. Prove that

$$\{g \in G \mid \pi(g) = 0\} \cup \{1\}$$

is a transitive normal subgroup of G.

(2) Let G be a non-abelian group of order pq, where p and q are distinct primes.
  (a) Prove that G is uniquely determined up to isomorphism.
  (b) Prove that G is a Frobenius group.
  (c) Find all irreducible characters of G.

(3) Let N be a normal subgroup of G and H a subgroup of G with $NH = G$ and $N \cap H = 1$. The following are equivalent.
  (a) $C_G(n) \subseteq N$ for every non-trivial $n \in N$.
  (b) $C_G(h) \subseteq H$ for every nontrivial $h \in H$.
  (c) Every $x \in G \setminus N$ is conjugate to an element of H.
  (d) Every non-trivial $h \in H$ is conjugate to every element of hN.
  (e) H is a Frobenius complement in G.

(4) Given an irreducible character χ of G, denote

$$\nu(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^2).$$

Let $t(G)$ be the number of involutions in G. Prove that

$$t(G) = -1 + \sum_{\chi \in \mathrm{Irr}(G)} \nu(\chi)\chi(1).$$

This homework is due 23 January 2014.