Vpisna številka:_____ Ime in priimek:_____

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(1) Sladoled vzamemo iz zamrzovalne skrinje s temperaturo $-20°C$ in ga odnesemo v sobo, katere temperatura je $25°C$. Po eni minuti se sladoled ogreje na $-10°C$. Po kolikšnem času se bo sladoled začel topiti (tj. se bo segrel na $0°C$)? (Temperatura sladoleda T se spreminja v skladu z diferencialno enačbo $\frac{dT}{dt} = k(T_0 - T)$, kjer je k konstanta, T_0 pa temperatura okolice.)

$T(t)$, $T(0) = -20$ $\quad T(1) = -10$ $\quad T(x) = 0$

$\frac{dT}{dt} = k(T_0 - T)$ $\quad T_0 = 25$

$\frac{dT}{T_0 - T} = k\,dt$

$-\ln(T_0 - T) = kt - \ln C$

$T_0 - T = Ce^{-kt}$

$T = T_0 - Ce^{-kt}$

$T(0) = 25 - C = -20 \Rightarrow C = +45$

$T = 25 + 45e^{-kt}$

$T(1) = 25 - 45e^{-k} = -10$

$45e^{-k} = 35$

$e^{-k} = \frac{7}{9}$

$-k = \ln\frac{7}{9}$

$T = 25 - 45e^{t\ln\frac{7}{9}}$

$T(x) = 25 - 45e^{x\ln\frac{7}{9}} = 0$

$e^{x\ln\frac{7}{9}} = \frac{5}{9}$

$x = \frac{\ln\frac{5}{9}}{\ln\frac{7}{9}} \doteq 2{,}33$ minute

(2) Preveri, da funkcija $y(x) = x$ reši enačbo

$$x^2y' + xy - y^2 = x^2$$

in nato poišči še splošno rešitev te enačbe.

$y = x$ $\quad x^2 \cdot 1 + x \cdot x - x^2 = x^2$ ✓

$y' = 1$

$y = x + \frac{1}{z}$

$y' = 1 - \frac{1}{z^2}z'$

$x^2 - \frac{x^2z'}{z^2} + x^2 + \frac{x}{z} - x^2 - \frac{1}{z^2} - \frac{2x}{z} = x^2$ $\quad / \cdot z^2$

$-x^2z' + xz - 1 - 2xz = 0$

$x^2z' + xz = -1$

H: $x^2z' + xz = 0$

$xz' + z = 0$

$\frac{z'}{z} = -\frac{1}{x}$

$\frac{dz}{z} = -\frac{dx}{x}$

$\ln z = -\ln x + \ln C$

$z_H = \frac{C}{x}$

P: $z_p = \frac{C(x)}{x}$

$z_p' = \frac{C'(x)\cdot x - C(x)}{x^2}$

$C'(x)\cdot x - C(x) + C(x) = -1$

$C'(x) = -\frac{1}{x}$

$C(x) = -\ln x$

$z_p = -\frac{\ln x}{x}$

$z = z_H + z_p = \frac{C - \ln x}{x}$

$y = x + \frac{x}{C - \ln x}$

(3) Preveri, da je enačba

$$((1+2\,\mathrm{sh}(x))y^2-(2\,\mathrm{ch}(x)+2x\,\mathrm{sh}(x))y+1)dx-(3y^2-2(x+2\,\mathrm{ch}(x))y+2x\,\mathrm{ch}(x)+1)dy=0$$

eksaktna in jo nato reši.

$f = (1+2\,\mathrm{sh}\,x)\,y^2-(2\,\mathrm{ch}\,x+2x\,\mathrm{sh}\,x)\,y+1$

$g = -(3y^2-2(x+2\,\mathrm{ch}\,x)y+2x\,\mathrm{ch}\,x+1)$

$f_y = 2y(1+2\,\mathrm{sh}\,x)-(2\,\mathrm{ch}\,x+2x\,\mathrm{sh}\,x)$

$g_x = 2y+2(\mathrm{sh}\,x)y-2\,\mathrm{ch}\,x-2x\,\mathrm{sh}\,x$ $\Big\}$ $\Rightarrow f_y = g_x$

enačba je eksaktna

$u_x = f,\ u_y = g$

$u_x = f \Rightarrow u = \int f\,dx = xy^2+2(\mathrm{ch}\,x)y^2-2(\mathrm{sh}\,x)y-2y\int x\,\mathrm{sh}\,x\,dx+x \Rightarrow$

$\int x\,\mathrm{sh}\,x\,dx = x\,\mathrm{ch}\,x-\int \mathrm{ch}\,x\,dx = x\,\mathrm{ch}\,x-\mathrm{sh}\,x$

$u = x \quad dv = \mathrm{sh}\,x\,dx$

$du = dx \quad v = \mathrm{ch}\,x$

$\Rightarrow u = xy^2+2y^2\,\mathrm{ch}\,x-2y\,\mathrm{sh}\,x-2yx\,\mathrm{ch}\,x+2y\,\mathrm{sh}\,x+x+C(y)$

$u_y = 2yx+4y\,\mathrm{ch}\,x-2x\,\mathrm{ch}\,x+C'(y)$

$= -3y^2+2xy+4y\,\mathrm{ch}\,x-2x\,\mathrm{ch}\,x-1$

$\Rightarrow C'(y) = -3y^2-1 \Rightarrow C(y) = -y^3-y+D$

$\Rightarrow u = xy^2+2y^2\,\mathrm{ch}\,x-2yx\,\mathrm{ch}\,x+x-y^3-y+D$

Rešitev: $xy^2+2y^2\,\mathrm{ch}\,x-2yx\,\mathrm{ch}\,x+x-y^3-y = E$

(4) Poišči splošno rešitev diferencialne enačbe

$$x^2y''-2xy'+2y=\frac{1+x}{x^2}.$$

H: $x^2y''-2xy'+2y=0$

$y = x^m$

$m(m-1)-2m+2=0$

$m^2-3m+2=0$

$(m-1)(m-2)=0$

$m_1 = 1$

$m_2 = 2$ $\quad y_H = Ax+Bx^2$

P: $y_P = y_{P_1}+y_{P_2}$

$y_{P_1} = Cx^{-2}$

$y_{P_1}' = -2Cx^{-3}$

$y_{P_1}'' = 6Cx^{-4}$

$6Cx^{-2}+4Cx^{-2}+2Cx^{-2} = x^{-2}$

$12C = 1$

$C = \frac{1}{12}$

$y_{P_2} = Dx^{-1}$

$y_{P_2}' = -Dx^{-2}$

$y_{P_2}'' = 2Dx^{-3}$

$2Dx^{-1}+2Dx^{-1}+2Dx^{-1} = x^{-1}$

$6D = 1$

$D = \frac{1}{6}$

$y_P = \frac{1}{12}x^{-2}+\frac{1}{6}x^{-1}$

$y = Ax+Bx^2+\frac{1}{12}x^{-2}+\frac{1}{6}x^{-1}$