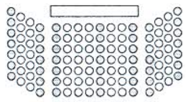
Matematika 1: 1. kolokvij

2. 12. 2013

Čas pisanja je 90 minut. Možno je doseči 100 točk. Veliko uspeha!



Sedež (2.05)

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Ime in priimek

Vpisna številka

1. naloga

S pomočjo matematične indukcije dokaži naslednji trditvi.

a) Za vsako naravno število $n \in \mathbb{N}$ velja

$$2 + 6 + 10 + \cdots + (4n - 2) = 2n^2.$$

b) Število $7^{2n-1} + 17^{n+1}$ je deljivo z 8 za vse $n \in \mathbb{N}$.

a) $n=1$: $4 \cdot 1 - 2 = 2 = 2 \cdot 1^2$ ✓

$n \to n+1$: $2 + 6 + 10 + \cdots + (4n-2) + (4(n+1)-2) =$

$= \big(2 + 6 + 10 + \cdots + (4n-2)\big) + (4n+2) \underset{\text{I.P.}}{=}$

$= 2n^2 + 4n + 2 = 2(n+1)^2$

b) $n=1$: $7 + 17^2 = 296 = 8 \cdot 37$ ✓

$n \to n+1$: $7^{2(n+1)-1} + 17^{(n+1)+1} = 7^{2n+1} + 17^{n+2} =$

$= 49 \cdot (7^{2n-1}) + 17 \cdot (17)^{n+1} =$

$= 48 \cdot 7^{2n-1} + 7^{2n-1} + 16 \cdot 17^{n+1} + 17^{n+1}$

$= \underbrace{(7^{2n-1} + 17^{n+1})}_{\text{deljivo z 8 po I.P.}} + \underbrace{8(6 \cdot 7^{2n-1} + 2 \cdot 17^{n+1})}_{\text{deljivo z 8.}}$

2. naloga

Poišči vsa kompleksna števila $z \in \mathbb{C}$, ki rešijo enačbo

$$|z|^2 - iz^2 - i\overline{z^2} = 2i|z|^2 - 4(i-1)$$

in jih skiciraj v kompleksni ravnini.

$z = a + ib$; $a, b \in \mathbb{R}$

$\bar{z} = a - ib$, $|z|^2 = a^2 + b^2$

$$(a^2+b^2) - i(a^2 + 2iab - b^2) - i(a^2 - 2iab - b^2) = 2i(a^2+b^2) - 4(i-1)$$

Realni del leve strani je enak realnem delu desne strani

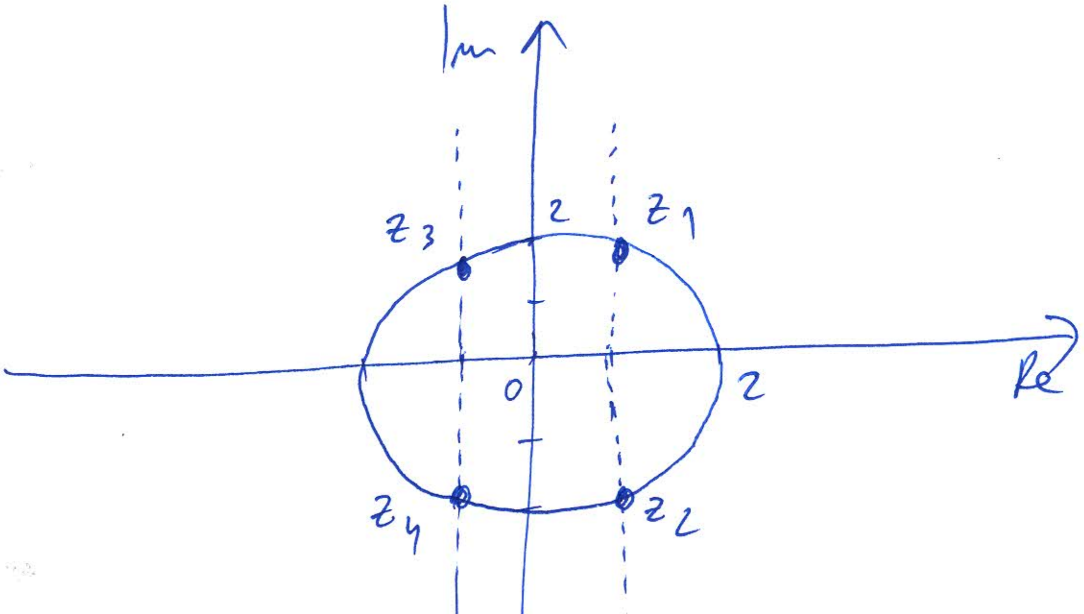
Imaginarni — " — Imaginaren — " —

$$a^2 + b^2 + 2ab - 2ab = 4 \Rightarrow a^2 + b^2 = 4$$

$$-a^2 + b^2 - a^2 + b^2 = 2a^2 + 2b^2 - 4 \Rightarrow 4a^2 = 4$$

$$\Rightarrow a = \pm 1 \quad b = \pm\sqrt{3}$$

$z_1 = 1 + i\sqrt{3}$, $z_2 = 1 - i\sqrt{3}$, $z_3 = -1 + i\sqrt{3}$, $z_4 = -1 - i\sqrt{3}$



3. naloga

Funkcija f je podana s predpisom

$$f(x) = \ln\left(1 - e^{\frac{x}{2}}\right).$$

a) Določi definicijsko območje funkcije f.

b) Dokaži, da je funkcija $f : \mathcal{D}_f \to \mathcal{Z}_f$ injektivna.

c) Določi inverzno funkcijo g k funkciji f.

d) Kaj je zaloga vrednosti funkcije f?

a) $D_f = \{x \in \mathbb{R} : 1 - e^{\frac{x}{2}} > 0\} = \{x \in \mathbb{R} : x < 0\} = (-\infty, 0)$

$1 - e^{\frac{x}{2}} > 0 \iff e^{\frac{x}{2}} < 1 \iff \frac{x}{2} < 0 \iff x < 0$

b) $f(x) = f(y) \Rightarrow x = y$

$\ln(1 - e^{\frac{x}{2}}) = \ln(1 - e^{\frac{y}{2}}) \quad / e$

$\Rightarrow 1 - e^{x/2} = 1 - e^{y/2} \Rightarrow e^{\frac{x}{2}} = e^{\frac{y}{2}} \Rightarrow \frac{x}{2} = \frac{y}{2} \Rightarrow x = y$

c) $x = \ln(1 - e^{\frac{y}{2}}) \Rightarrow e^x = 1 - e^{y/2} \Rightarrow e^{y/2} = 1 - e^x \Rightarrow$

$\Rightarrow y/2 = \ln(1 - e^x) \Rightarrow y = 2\ln(1 - e^x)$

$\Rightarrow g(x) = 2\ln(1 - e^x)$

d) $Z_f = D_g = \{x \in \mathbb{R} : 1 - e^x > 0\} = (-\infty, 0)$

4. naloga

a) Izračunaj limito

$$\lim_{n\to\infty}(\sqrt{n^2+4n}-\sqrt{n^2-2n}).$$

b) Izračunaj limito

$$\lim_{n\to\infty}\left(\frac{n-4}{n+3}\right)^{\frac{n^3}{2n^2+1}}.$$

c) Zaporedje $\{a_n\}_{n\in\mathbb{N}}$ je podano z rekurzivno zvezo

$$a_{n+1}=\sqrt{a_n+2}$$

in prvim členom $a_1=\frac{1}{2}$. Dokaži, da je zaporedje $\{a_n\}_{n\in\mathbb{N}}$ konvergentno in izračunaj njegovo limito.

a) $$\lim_{n\to\infty}\left(\sqrt{n^2+4n}-\sqrt{n^2-2n}\right)=\lim_{n\to\infty}\frac{(n^2+4n)-(n^2-2n)}{\sqrt{n^2+4n}+\sqrt{n^2-2n}}=$$

$$=\lim_{n\to\infty}\frac{6n}{\sqrt{n^2+4n}+\sqrt{n^2-2n}}=\lim_{n\to\infty}\frac{6}{\sqrt{1+\frac{4}{n}}+\sqrt{1-\frac{2}{n}}}=\frac{6}{2}=\underline{\underline{3}}$$

b) $$\lim_{n\to\infty}\left(\frac{n-4}{n+3}\right)^{\frac{n^3}{2n^2+1}}=\lim_{n\to\infty}\left(1+\frac{1}{-\frac{n+3}{7}}\right)^{\frac{n^3}{2n^2+1}}=$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{-\frac{n+3}{7}}\right)^{-\frac{n+3}{7}\cdot\left(-\frac{7}{n+3}\cdot\frac{n^3}{2n^2+1}\right)}=$$

$$=e^{\lim\limits_{n\to\infty}-\frac{7n^3}{(n+3)(2n^2+1)}}=e^{-\lim\limits_{n\to\infty}\frac{7}{(1+\frac{3}{n})(2+\frac{1}{n^2})}}=\underline{\underline{e^{-\frac{7}{2}}}}$$

c) $a_1=\frac{1}{2}$, $a_2=\sqrt{\frac{5}{2}}>a_1$.

$a_{n+1}>a_n \quad \forall n\in\mathbb{N}$.

$n=1$ ✓

$n\to n+1$: $a_{n+2}-a_{n+1}=\sqrt{a_{n+1}+2}-\sqrt{a_n+2}=\dfrac{a_{n+1}-a_n}{\sqrt{a_{n+1}+2}+\sqrt{a_n+2}}$

$\sqrt{a_n+2} > 0 \quad \forall n \in \mathbb{N}$, (ker $a_n > 0 \to$ dobra def. $\sqrt{a_n+2}$)

$a_{n+2} - a_{n+1}$ je produkt dveh poz. št. $\to a_{n+2} > a_{n+1}$. ✓.

$\{a_n\}_{n\in\mathbb{N}}$ navzgor omejeno z 2

$n=1: \quad a_1 = \frac{1}{2} < 2$ ✓

$n \to n+1 \quad a_{n+1} = \sqrt{a_n+2} \leq \sqrt{2+2} = \sqrt{4} = 2$ ✓

$\{a_n\}_{n\in\mathbb{N}}$ je naraščajoče navzgor omejeno $\Rightarrow$

$\exists \lim_{n\to\infty} a_n =: a$

$$a_{n+1} = \sqrt{a_n+2}$$

$$\Big\downarrow \lim_{n\to\infty}$$

$$a = \sqrt{a+2} \Rightarrow a^2 = a+2$$

$$\Rightarrow a^2 - a - 2 = 0$$

$$(a-2)(a+1) = 0$$

$a = 2$ ali $a = -1$

Primer -1 odpade, ker $\forall n \in \mathbb{N}$ $a_n \geq 0$.

Torej $\lim_{n\to\infty} a_n = \underline{\underline{2}}$.