1) a) $$\lim_{n\to\infty} \frac{n^2\sqrt{n}+2n-7}{3(n-1)(n+2)^{3/2}+2n} \overset{:n^{5/2}}{=} \lim_{n\to\infty} \frac{1+\frac{2}{n^{3/2}}-\frac{7}{n^{5/2}}}{3\left(1-\frac{1}{n}\right)\cdot\left(1+\frac{2}{n}\right)^{3/2}+\frac{2}{n^{3/2}}}$$

$$= \frac{1+0-0}{3\cdot1\cdot1+0} = \underline{\underline{\frac{1}{3}}}$$

b) $$\lim_{n\to\infty}\left(\sqrt{n^2+4n+8}-n\right) = \lim_{n\to\infty} \frac{\left(\sqrt{n^2-4n+8}-n\right)\left(\sqrt{n^2-4n+8}+n\right)}{\sqrt{n^2-4n+8}+n} =$$

$$= \lim_{n\to\infty} \frac{(n^2-4n+8)-n^2}{\sqrt{n^2-4n+8}+n} = \lim_{n\to\infty} \frac{-4n+8}{\sqrt{n^2-4n+8}+n} =$$

$$= \lim_{n\to\infty} \frac{-4+\frac{8}{n}}{\sqrt{1-\frac{4}{n}+\frac{8}{n^2}}+1} = -\frac{4}{1+1} = \underline{\underline{-2}}$$

c) $$\lim_{n\to\infty}\left(n+\sqrt[3]{1-n^3}\right) = \lim_{n\to\infty} \frac{\left(n+\sqrt[3]{1-n^3}\right)\left(n^2-n\sqrt[3]{1-n^3}+\sqrt[3]{(1-n^3)^2}\right)}{n^3-n\sqrt[3]{1-n^3}+\sqrt[3]{(1-n^3)^2}} =$$

$$= \lim_{n\to\infty} \frac{n^3+(1-n^3)}{n^3-n\sqrt[3]{1-n^3}+\sqrt[3]{(1-n^3)^2}} = \underline{\underline{0}}$$

d) $$\lim_{n\to\infty} \frac{2^{3n+1}-3^{n+2}-6^n}{2^n-9^n+8^{n+2}} = \lim_{n\to\infty} \frac{2\cdot8^n-9\cdot3^n-6^n}{2^n-9^n+64\cdot8^n} =$$

$$= \lim_{n\to\infty} \frac{2\cdot\left(\frac{8}{9}\right)^n-9\cdot\left(\frac{3}{9}\right)^n-\left(\frac{6}{9}\right)^n}{\left(\frac{2}{9}\right)^n-1+64\cdot\left(\frac{8}{9}\right)^n} = \frac{2\cdot0-9\cdot0-0}{0-1+64\cdot0} = \underline{\underline{0}}$$

2) a) $\lim\limits_{n\to\infty} \sqrt[3n]{17} = \lim\limits_{n\to\infty} \sqrt[n]{\underbrace{(17)^{1/3}}_{c}} = \lim\limits_{n\to\infty} \sqrt[n]{c} = 1$

b)

$1 \le n\sqrt{n+2}$ in

$$1 \le \sqrt[4n]{n\sqrt{n+2}} = \sqrt[4n]{n} \cdot \sqrt[4n]{\sqrt{n+2}}$$
$$= \sqrt[4n]{n} \cdot \sqrt[8n]{n+2}$$

Za $n \in \mathbb{N}$:

$$1 \le n \le 4n \Rightarrow 1 \le \sqrt[4n]{n} \le \sqrt[4n]{4n}$$

$1 \xrightarrow{n\to\infty} 1$, $\sqrt[4n]{n} \xrightarrow{n\to\infty} 1$, $\sqrt[4n]{4n} \xrightarrow{n\to\infty} 1$

in

$$1 \le n+2 \le 8n \Rightarrow 1 \le \sqrt[8n]{n+2} \le \sqrt[8n]{8n}.$$

$1 \xrightarrow{n\to\infty} 1$, $\sqrt[8n]{n+2} \xrightarrow{n\to\infty} 1$, $\sqrt[8n]{8n} \xrightarrow{n\to\infty} 1$

Po izreku o sendvič

je $\lim\limits_{n\to\infty} \sqrt[4n]{n\sqrt{n+2}} = 1$

c) $\lim\limits_{n\to\infty} \left(\frac{2n-1}{2n+1}\right)^{n+2} = \lim\limits_{n\to\infty} \left(1 + \frac{1}{\frac{2n+1}{-2}}\right)^{n+2} =$

$= \lim\limits_{n\to\infty} \left(1 + \frac{1}{\frac{2n+1}{-2}}\right)^{\frac{2n+1}{-2} \cdot \left(\frac{n+2}{2n+1} \cdot (-2)\right)}$ $\quad m = \frac{2n+1}{2}$

$= \lim\limits_{n\to\infty} \left(1 - \frac{1}{\frac{2n+1}{2}}\right)^{-\frac{2n+1}{2} \cdot \frac{-2(n+2)}{2n+1}} = \lim\limits_{m\to\infty} \left(1 - \frac{1}{m}\right)^{-m \cdot a} =$

$a = \frac{2m-3}{2m}$

$$e^{\lim_{n\to\infty}\left(-\frac{2n-3}{2n}\right)} = e^{-1}$$

d) $\lim_{n\to\infty}\left(\frac{n-3}{2n+2}\right)^{2n-1}$

ker $0 \le \frac{n-3}{2n+2} \le \frac{1}{2} \Rightarrow 0 \le \left(\frac{n-3}{2n+2}\right)^{2n-1} \le \left(\frac{1}{2}\right)^{2n-1}$

$$\underset{0}{\downarrow}{\scriptstyle n\to\infty} \qquad \underset{0}{\downarrow}{\scriptstyle n\to\infty} \qquad \underset{0}{\downarrow}{\scriptstyle n\to\infty}$$

je limita enaka 0.

e) $$\lim_{n\to\infty}\left(\frac{n^2+2n}{n^2+1}\right)^{3n} = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2+1}{2n-1}}\right)^{3n} =$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{\frac{n^2+1}{2n-1}}\right)^{\frac{2n-1}{n^2+1}\cdot\frac{n^2+1}{2n-1}\cdot 3n} =$$

$$= e^{\lim_{n\to\infty}\frac{3n(2n-1)}{n^2+1}} = \underline{\underline{e^6}}$$

3) $$a_{n+1} = \left((n+1)a_n^n\right)^{\frac{1}{n+1}}$$

$a_1 = 1$, $a_2 = (2a_1)^{\frac{1}{2}} = 2^{1/2}$

$a_3 = (3a_2^2)^{1/3} = (3\cdot 2)^{1/3}$

$a_4 = (4a_3^3)^{1/4} = (4\cdot 3\cdot 2)^{1/4} = \sqrt[4]{4!}$

$\vdots$

Ugenemo $a_n = \sqrt[n]{n!}$

$n=1$ ✓

$n \to n+1$:

$$a_{n+1} = \left((n+1)\cdot a_n^n\right)^{\frac{1}{n+1}} \underset{\text{I.P.}}{=} \left((n+1)\cdot\left(\sqrt[n]{n!}\right)^n\right)^{\frac{1}{n+1}}$$

$$= \left((n+1)\cdot n!\right)^{\frac{1}{n+1}} = \underline{\left((n+1)!\right)^{\frac{1}{n+1}}}$$

<u>dokažemo monotono</u>

$$a_{n+1} > a_n \Leftrightarrow \sqrt[n+1]{(n+1)!} > \sqrt[n]{n!} \Leftrightarrow \left((n+1)!\right)^n > (n!)^{n+1}$$

$$\Leftrightarrow \left(\frac{(n+1)!}{n!}\right)^n > n! \Leftrightarrow (n+1)^n \overset{?}{>} n!$$

velja $n! = 1\cdot 2\cdot \ldots \cdot (n-1)\cdot n < (n+1)\cdot(n+1)\cdot \ldots \cdot (n+1)\cdot(n+1)$
$= (n+1)^n$ ✓

4) $a_{n+1} = \dfrac{a_n^2 + a_n}{2}$

$a_1 \in [0,1]$

$a_2 = \dfrac{a_1^2 + a_1}{2} \leq \dfrac{a_1 + a_1}{2} = a_1$

$a_1 > 1$

$a_2 = \dfrac{a_1^2 + a_1}{2} > \dfrac{a_1 + a_1}{2} = a_1$

$a_{n+2} - a_{n+1} = \frac{1}{2}\cdot\left(a_{n+1}^2 + a_{n+1} - a_n^2 - a_n\right) = \frac{1}{2}\left(a_{n+1} - a_n\right)\left(a_{n+1} + a_n + 1\right)$

očitno, če $a_n \geq 0 \Rightarrow a_{n+1} \geq 0$ ✓

<u>za $a_1 \in [0,1]$ zap. pada</u>

od zgoraj vemo $a_2 < a_1$

<u>za $a_1 > 1$ zap. raste</u>

$a_1 < a_2$

Hkrati:

$a_{n+2} - a_{n+1} = \frac{1}{2}\underbrace{(a_n + a_{n+1} + 1)}_{>0}\cdot\underbrace{(a_{n+1} - a_n)}_{\text{bodisi pozitivno / bodisi negativno}}$ (odvisno od I.P.)

4

Če $a_1 \in [0,1] \Rightarrow \{a_n\}_{n\in\mathbb{N}}$ pada in je navzdol omejeno z 0.

$\Rightarrow \{a_n\}_{n\in\mathbb{N}}$ konv. $\Rightarrow$ $L = \frac{L^2+L}{2} \Rightarrow L^2 - L = 0$

$\Rightarrow L = 0$ ali $L = 1$.

Če $a_1 < 1 \Rightarrow L = 0$

Če $a_1 = 1 \Rightarrow L = 1$, saj

$a_1 = 1 \Rightarrow a_2 = \frac{1+1}{2} = 1$

$\underline{a_n = 1 \quad \forall n \in \mathbb{N}}$

$n \to n+1 \qquad a_{n+1} = \frac{a_n^2 + a_n}{2} = \frac{1+1}{2} = 1$

Če $a_1 > 1 \Rightarrow$ zap. naraša. Če limita obstaja, je večja od 1. Edina kandidata sta 0 in 1

$\cancel{}$.

5) $\displaystyle\sum_{n=1}^{\infty} \frac{(n+3)(n+2)n! - (n+2)!}{(n+3)!} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+3}\right)$

$s_k = \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \dots + \frac{1}{k+1} - \frac{1}{k+3}\right)$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{k+2} - \frac{1}{k+3}$

$\lim\limits_{k\to\infty} s_k = \frac{1}{2} + \frac{1}{3} = \underline{\underline{\frac{5}{6}}}$

6) a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(4n+4)^n}$ kor. krit. $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \frac{n}{4(n+1)} = \frac{1}{4}$

Konv. po kor. krit.

b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{n^{2n}}$ kvoc. krit.: $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\frac{2^{n+1}(n+1)!}{(n+1)^{2(n+1)}}}{\frac{2^n \cdot n!}{n^{2n}}} =$

$$\lim_{n\to\infty} \frac{2\cdot(n+1)\cdot n^{2n}}{(n+1)^{2(n+1)}} = \lim_{n\to\infty} \frac{2\cdot n^{2n}}{(n+1)^{2n}\cdot(n+1)} =$$

$$= \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^{2n}\cdot(n+1)} = \underline{\underline{0}}$$

vrsta konv. po kvoc. krit.

c) $\sum_{n=1}^{\infty} \frac{1}{\cos\frac{\pi}{2n}}$ $\qquad \lim_{n\to\infty} \frac{1}{\cos\frac{\pi}{2n}} = \frac{1}{\cos 0} = \underline{\underline{1}}$

vrsta div.

d) $\sum_{n=1}^{\infty} \left(\frac{1+n}{1+n^2}\right)^2$; $\qquad \left(\frac{1+n}{1+n^2}\right)^2 \sim \left(\frac{1}{n}\right)^2$

Primerjalni kriterij

$$\left(\frac{1+n}{1+n^2}\right)^2 \overset{?}{\leq} \frac{1}{n^2}$$

$$\frac{1}{n^2} - \left(\frac{1+n}{1+n^2}\right)^2 = \frac{(1+n^2)^2 - (1+n)^2 n^2}{n^2(1+n^2)} = \frac{(1+2n^2+n^4) - (1+2n+n^2)n^2}{n^2(1+n^2)}$$

$$= 1+n^2-2n^3$$

- NE

$$\left(\frac{1+n}{1+n^2}\right)^2 \overset{?}{\leq} \frac{2}{n^2} \ :$$

$$\frac{2}{n^2} - \left(\frac{1+n}{1+n^2}\right)^2 = \frac{2(1+2n^2+n^4) - (1+2n+n^2)n^2}{n^2(1+n^2)} =$$

$$= \frac{n^4+3n^2-2n^3+2}{n^2(1+n^2)} = \frac{n^2(n^2-2n+3)+2}{n^2(1+n^2)} \underline{\underline{> 0}}$$

Po primerj. krit. z vrsto $\sum_{n=1}^{\infty} \frac{2}{n^2}$ vrsta konv.

7) a) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n n}{n^2+2}$ konv. po Leibnizu:

$\lim\limits_{n\to\infty} \frac{n}{n^2+2} = 0$;

$a_n - a_{n+1} = \frac{n}{n^2+2} - \frac{n+1}{(n+1)^2+2} = \frac{n((n+1)^2+2) - (n^2+2)(n+1)}{(n^2+2)((n+1)^2+2)} =$

$= \frac{(n\cdot(n^2+2n+3) - (n^3+2n+n^2+2))}{(n^2+2)((n+1)^2+2)} =$

$= \frac{n^2+n-2}{(n^2+2)((n+1)^2+2)} \geq 0 \quad \forall n \in \mathbb{N}.$

Po Leibnizovem kriteriju vrsta konv.

vrsta div. abs

$\sum\limits_{n=1}^{\infty} \frac{n}{n^2+2}$ $\qquad 0 \leq \frac{1}{n} \leq \frac{10n}{n^2+2}$

$\frac{10n}{n^2+2} - \frac{1}{n} = \frac{10n^2-1}{n(n^2+2)} > 0.$

Po prim. krit. vrsta div.

b) $\sum\limits_{n=1}^{\infty} \frac{\sin 7n}{(\ln 4)^n}$ konv. abs.

$\sum\limits_{n=1}^{\infty} \left|\frac{\sin 7n}{(\ln 4)^n}\right|$ $\qquad \left|\frac{\sin 7n}{(\ln 4)^n}\right| \leq \frac{1}{(\ln 4)^n}$

$\ln 4 > \ln e = 1$ $\qquad \sum\limits_{n=1}^{\infty} \frac{1}{(\ln 4)^n}$ je geom. vrsta in konv., ker $q < 1$.

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Po prim. krit. $\sum_{n=1}^{\infty} \left|\frac{\sin 7n}{(\ln^{4n} n)^n}\right|$ konv.

Ker vsaka abs. konv. vrsta konv. $\Rightarrow$ vrsta iz naloge konv.

8) a) $f(x) = \frac{x^3 - 3x + 2}{x^2 - 9}$

Ničle: $x^3 - 3x + 2 = 0$

~~x=1~~; $x=1$:

	1	0	-3	2
1		1	1	-2
	1	1	-2	0

$x^3 - 3x + 2 = (x-1)(x^2 + x - 2)$
$= (x-1)(x+2)(x-1)$

$x_1 = 1 \quad x_3 = -2$
$x_2 = 1$

Poli $x_1 = -3, \quad x_2 = 3$

Z.v. $f(0) = -\frac{2}{9}$. ~~asimptota: $(x^3 - 3x + 2) : (x^2 - 9) = x$~~

asimptota:

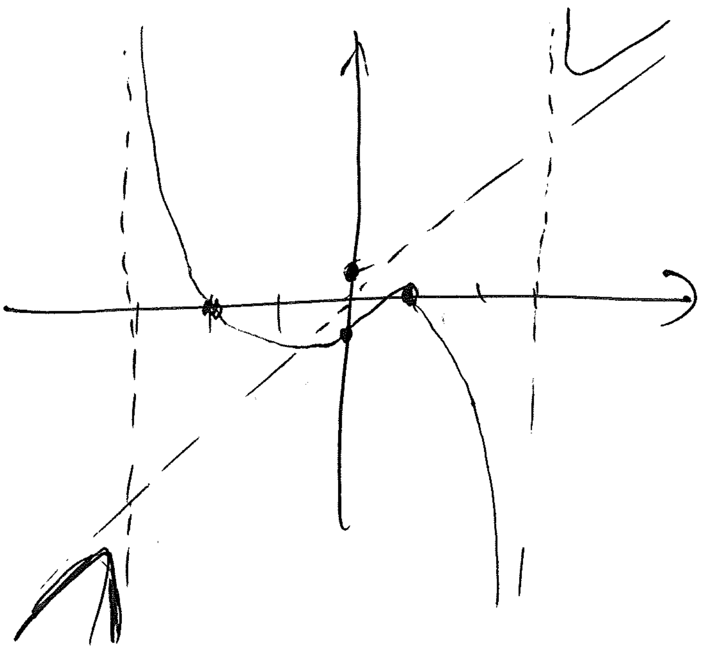
$(x^3 - 3x + 2) : (x^2 - 9) = x + \frac{6x + 2}{x^2 - 9}$

$-x^3 + 9x$

$6x + 2$

Presečišče z asimptoto:

$f(x) = x \Rightarrow \frac{6x+2}{x^2-9} = 0 \Rightarrow x = -\frac{1}{3}$



b) $g(x) = ax^2 + bx + c$

$g(-1/3) = -1/3 \Rightarrow -1/3 = \frac{a}{9} - \frac{b}{3} + c \Rightarrow -\frac{1}{3} = \frac{a}{9} - \frac{b}{3} - \frac{2}{9} \Big/ \cdot 9$

$g(0) = -2/9 \Rightarrow -\frac{2}{9} = c$

$-3 = a - 3b - 2$

$a = 3b - 1$

$g(x) = (3b-1)x^2 + bx - \frac{2}{9}$

c) x Koordinaata tenera jö $\frac{x_1 + x_2}{2}$

$x_{1,2} = \frac{-b \pm \sqrt{b^2 + \frac{8}{9}(3b-1)}}{2(3b-1)}$

$\Rightarrow x = \frac{b}{2(1-3b)} \Rightarrow b = \frac{2x}{1+6x}$

$y = (3b-1) \cdot \frac{b^2}{4 \cdot (3b-1)^2} - \frac{b^2}{2(3b-1)} - \frac{2}{9} = \frac{b^2}{4(3b-1)} - \frac{b^2}{2(3b-1)} - \frac{2}{9} =$

$= -\frac{b^2}{4(3b-1)} - \frac{2}{9}$

$\Rightarrow y = -\frac{4x^2}{(1+6x)^2} \cdot \frac{1}{4 \cdot \left(\frac{6x}{1+6x} - 1\right)} - \frac{2}{9} = -\frac{x^2}{(1+6x)^2} \cdot \frac{1+6x}{-1} - \frac{2}{9} =$

$= +\frac{x^2}{1+6x} - \frac{2}{9} = \frac{9x^2 - 12x - 2}{9(1+6x)}$

Nicla: $9x^2 - 12x - 2 = 0$

$x_{1,2} = \frac{12 \pm \sqrt{144 + 72}}{18} = \frac{12 \pm \sqrt{216}}{18} = \frac{2 \pm \sqrt{6}}{3}$

$x_1 \approx 1{,}48$

$x_2 \approx -0{,}15$

9

Pol.: $x = -\frac{1}{6}$

Zač. vr. $y(0) = -\frac{2}{9}$

asimpt. $(9x^2-12x-2):(54x+9) = \frac{x}{6} - \frac{1}{4} + \frac{9}{4(54x+9)}$

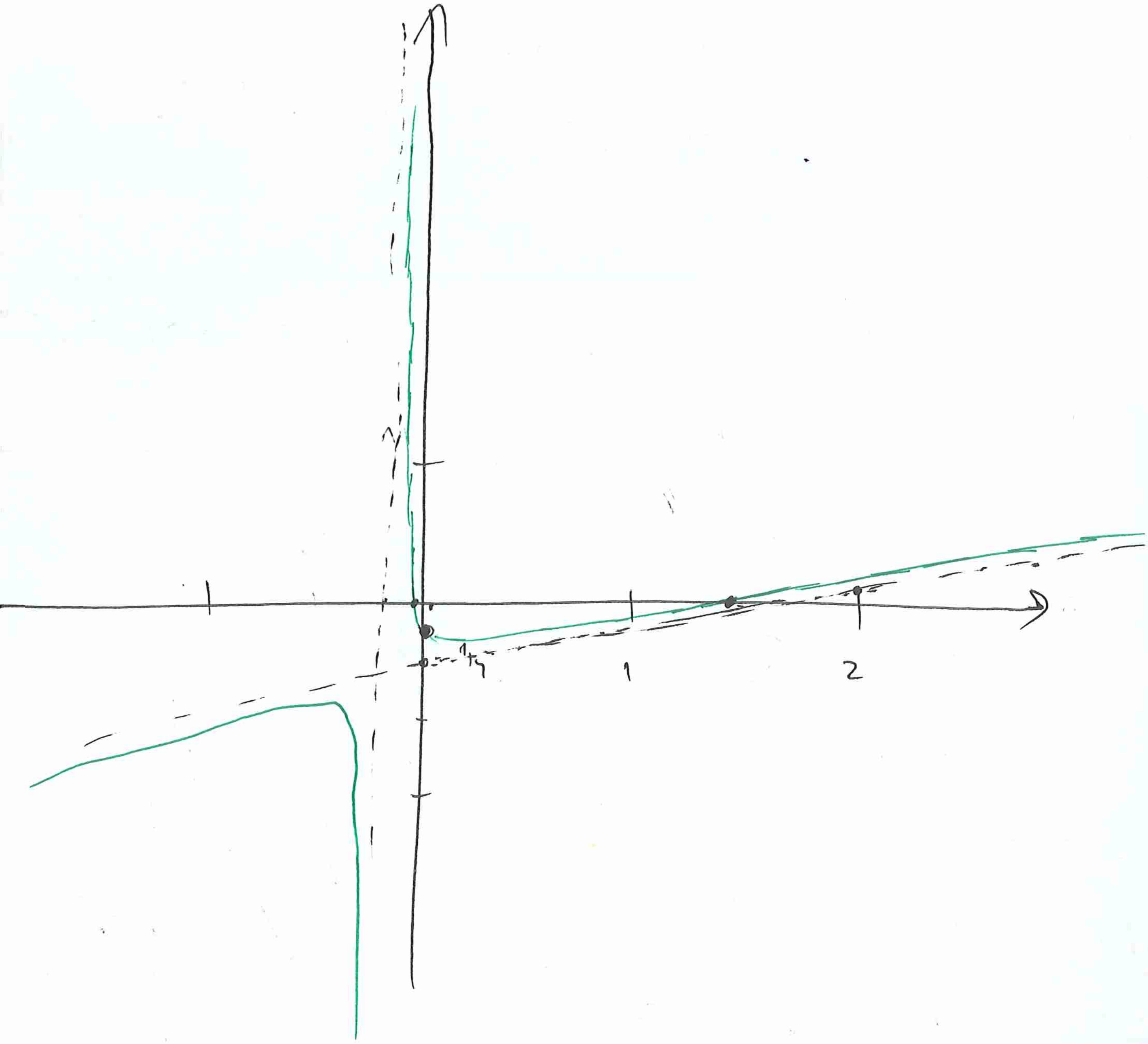
$-9x^2 - \frac{3}{2}x$

$-\frac{27}{2}x - 2$

$+\frac{27}{2}x + \frac{9}{3}$

$\frac{9}{4}$

$= \frac{x}{6} - \frac{1}{4} + \frac{1}{36(1+6x)}$



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9) a) ~~D_f: ...~~

D_f: $\ln x \neq 1$

$x > 0$; $\Rightarrow D_f = (0,\infty) \setminus \{e\}$

Z_f: $\dfrac{1+\ln x}{1-\ln x} = y \Rightarrow 1 + \ln x = (1-\ln x)y$

$\ln(x) \cdot (1+y) = y-1$

$\ln x = \dfrac{y-1}{y+1}$

$x = e^{\frac{y-1}{y+1}}$; $Z_f = \mathbb{R} \setminus \{-1\}$.

b) $f(x) = f(y)$

$\dfrac{1+\ln x}{1-\ln x} = \dfrac{1+\ln y}{1-\ln y} \Rightarrow \cancel{1} + \ln x - \ln y - \cancel{\ln x \ln y} = \cancel{1} + \ln y - \ln x - \cancel{\ln x \ln y}$

$\Rightarrow 2\ln x = 2\ln y$

$\Rightarrow \ln x = \ln y \Rightarrow \boxed{x = y}$

$f: (0,\infty) \setminus \{e\} \to \mathbb{R} \setminus \{-1\}$ je bij

c) $f(y) = e^{\frac{y-1}{y+1}}$ $\quad \dfrac{1+x}{1-x}$

d)