Rešitve izpita iz Matematike 2 z dne 20. 8. 2012

Praktična matematika

1. Funkcija d ni metrika na $\mathbb{N}$, ker ne velja trikotniška neenakost: velja $d(1,2) = 1/2$, medtem ko je $d(1,5) + d(5,2) = 1/5 + 1/5 = 2/5 < 1/2$.

2. Ko v dano zvezo:

$$e^{2x+yz} + (1+x)(2+y)z = 0 \tag{1}$$

vstavimo $x = 0$ in $y = 0$, dobimo $z = -1/2$. Nadalje, ko zvezo (1) parcialno odvajamo po x, dobimo:

$$e^{2x+yz}\left(2 + y\,\frac{\partial z}{\partial x}\right) + (2+y)z + (1+x)(2+y)\frac{\partial z}{\partial x} = 0 \tag{2}$$

in ko sem vstavimo $x = 0$, $y = 0$ in $z = -1/2$, dobimo $\frac{\partial z}{\partial x} = -1/2$. Ko pa zvezo (1) parcialno odvajamo po y, dobimo:

$$e^{2x+yz}\left(z + y\,\frac{\partial z}{\partial y}\right) + (1+x)z + (1+x)(2+y)\frac{\partial z}{\partial y} = 0$$

in ko spet vstavimo $x = 0$, $y = 0$ in $z = -1/2$, dobimo $\frac{\partial y}{\partial x} = 1/2$. Končno še zvezo (2) parcialno odvajamo po x. Dobimo:

$$e^{2x+yz}\left(2 + y\,\frac{\partial z}{\partial x}\right)^2 + e^{2x+yz}\,y\,\frac{\partial^2 z}{\partial z^2} + 2(2+y)\frac{\partial z}{\partial x} + (1+x)(2+y)\frac{\partial^2 z}{\partial x^2} = 0$$

in ko vstavimo $x = 0$, $y = 0$, $z = -1/2$ in $\frac{\partial z}{\partial x} = -1/2$, dobimo še $\frac{\partial^2 z}{\partial x^2} = -1$.

3. Velja:

$$F'(a) = \frac{2a}{\ln(a+a^2)} - \frac{1}{\ln(2a)} - \int_a^{a^2} \frac{\mathrm{d}x}{(a+x)\big(\ln(a+x)\big)^2}\,,$$

od koder dobimo:

$$F'(1) = \frac{2}{\ln 2} - \frac{1}{\ln 2} - \int_1^1 \frac{\mathrm{d}x}{(a+x)\big(\ln(a+x)\big)^2} = \frac{1}{\ln 2}\,.$$

4. Označimo iskani integral z J. Po prevedbi na cilindrične koordinate:

$$x = r\cos\theta\,, \quad y = r\sin\theta\,, \quad z = z\,; \quad J = r$$

dobimo:

$$J = \iiint_{\substack{r \ge 0 \\ 0 \le \theta \le 2\pi \\ r+z \le 2}} r^3 z^2 \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta\,.$$

Od tod naprej gre na vsaj dva smiselna, bolj ali manj ekvivalentna načina.

Prvi način. Iz:

$$J = 2\pi \int_0^2 \int_0^{2-r} r^3 z^2 \,\mathrm{d}z\,\mathrm{d}r = \frac{2\pi}{3} \int_0^2 r^3 (2-r)^2 \,\mathrm{d}r$$

s substitucijo $r = 2t$ dobimo:

$$J = \frac{256\pi}{3} \int_0^1 t^3 (1-t)^3 \,\mathrm{d}t = \frac{256\pi}{3} \,\mathrm{B}(4,4) = \frac{256\pi}{3} \frac{\big(\Gamma(4)\big)^2}{\Gamma(8)} = \frac{256\pi}{3} \frac{(3!)^2}{7!} = \frac{64\pi}{105}\,.$$

Drugi način. Iz:

$$J = 2\pi \int_0^2 \int_0^{2-z} r^3 z^2 \,\mathrm{d}r\,\mathrm{d}z = \frac{\pi}{2} \int_0^2 (1-z)^4 z^2 \,\mathrm{d}r$$

s substitucijo $z = 2t$ dobimo:

$$J = 64\pi \int_0^1 t^2 (1-t)^4 \,\mathrm{d}t = 64\pi \,\mathrm{B}(5,3) = 64\pi \frac{\Gamma(5)\,\Gamma(3)}{\Gamma(8)} = 64\pi \frac{4!\,2!}{7!} = \frac{64\pi}{105}\,.$$

5. Substitucija $z = e^{it}$, $\cos t = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$, $\sin t = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)$, $\mathrm{d}t = \dfrac{\mathrm{d}z}{iz}$, nam da:

$$I := \int_0^{2\pi} \frac{1 + \sin t}{2 + \cos t} \,\mathrm{d}t = \oint_K f(z)\,\mathrm{d}z\,,$$

kjer je K pozitivno orientirana enotska krožnica, funkcija f pa je definirana po predpisu:

$$f(z) = \frac{-z^2 - 2iz + 1}{z(z^2 + 4z + 1)} = \frac{-z^2 - 2iz + 1}{z\big(z + 2 - \sqrt{3}\big)\big(z + 2 + \sqrt{3}\big)}\,.$$

Funkcija f ima tri singularnosti: 0, $-2 + \sqrt{3}$ in $-2 - \sqrt{3}$. Le prvi dve ležita znotraj enotskega kroga, zato velja:

$$I = 2\pi i \Big[\mathrm{Res}(f, 0) + \mathrm{Res}\big(f, -2 + \sqrt{3}\big)\,.$$

Ker gre obakrat za pol prve stopnje, je:

$$\begin{aligned}
\mathrm{Res}(f, 0) &= \lim_{z \to 0} z f(z) = \left.\frac{-z^2 - 2iz + 1}{z^2 + 4z + 1}\right|_{z=0} = 1\,,\\
\mathrm{Res}\big(f, -2 + \sqrt{3}\big) &= \lim_{z \to -2+\sqrt{3}} \big(z + 2 - \sqrt{3}\big) f(z) =\\
&= \left.\frac{-z^2 - 2iz + 1}{z\big(z + 2 + \sqrt{3}\big)}\right|_{z=-2+\sqrt{3}} =\\
&= \frac{6 - 4\sqrt{3} - 2\big(2 - \sqrt{3}\big)i}{2\sqrt{3}\big(2 - \sqrt{3}\big)} =\\
&= -1 - \frac{i}{\sqrt{3}}\,.
\end{aligned}$$

Sledi $I = \dfrac{2\pi}{\sqrt{3}} = \dfrac{2\pi\sqrt{3}}{3}$.

Opomba. V integral $\displaystyle\int_0^{2\pi} \frac{\sin t}{2+\cos t}\,\mathrm{d}t$ lahko povsem legitimno uvedemo substitucijo $x = \cos t$ in dobimo $\displaystyle -\int_1^1 \frac{\mathrm{d}x}{2+x} = 0$. Zato bi bilo dovolj izračunati le $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}t}{2+\cos t}$.