Rešitve izpita iz Matematike 2 z dne 29. 8. 2013

Praktična matematika

1. Iz:

$$\begin{aligned}\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x &= \frac{2}{\pi}\int_0^{\pi/2}\Big(\frac{\pi}{2}-x\Big)\sin(nx)\,\mathrm{d}x = \\ &= \frac{1}{n}-\frac{2}{\pi n^2}\sin\frac{n\pi}{2} = \\ &= \begin{cases} \frac{1}{n}-\frac{2}{\pi n^2} & ;\ n=1,5,9,13,\ldots \\ \frac{1}{n} & ;\ n=2,4,6,8,10,\ldots \\ \frac{1}{n}+\frac{2}{\pi n^2} & ;\ n=3,7,11,15,\ldots \end{cases}\end{aligned}$$

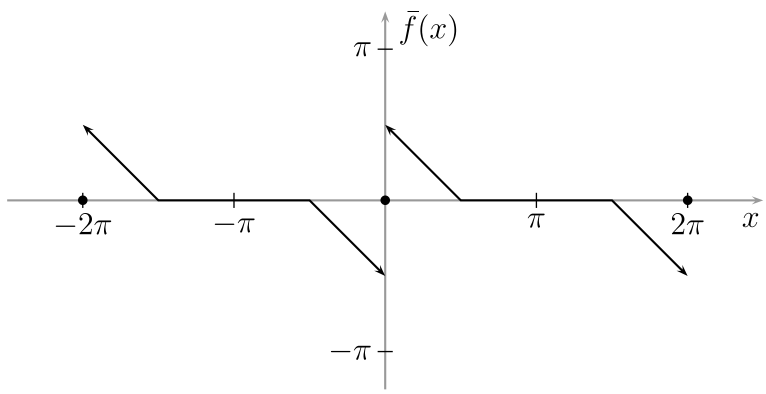
dobimo:

$$\begin{aligned}\bar f(x) = \sin x + \frac{\sin(2x)}{2}+\frac{\sin(3x)}{3}+\frac{\sin(4x)}{4}+\cdots - \\ -\frac{2}{\pi}\left(\sin x - \frac{\sin(3x)}{3^2}+\frac{\sin(5x)}{5^2}-\frac{\sin(7x)}{7^2}+\cdots\right),\end{aligned}$$

kjer je $\bar f$ periodična s periodo 2π in:

$$\bar f(x) = \begin{cases} 0 & ;\ -\pi \le x \le -\frac{\pi}{2} \\ -\frac{\pi}{2}-x & ;\ -\frac{\pi}{2} \le x < 0 \\ 0 & ;\ x = 0 \\ \frac{\pi}{2}-x & ;\ 0 < x \le \frac{\pi}{2} \\ 0 & ;\ \frac{\pi}{2} \le x \le \pi \end{cases}.$$

Graf:



2. $T_2(x,y) = 2x^2 + 2y$.

3. Najprej krivuljo parametriziramo z z:

$$\vec{r} = \begin{bmatrix} -\frac{1}{z^2} + z^2 \\ \frac{1}{z} \\ z \end{bmatrix}.$$

Če s piko označimo odvod po z, dobimo:

$$\dot{\vec{r}} = \begin{bmatrix} \frac{2}{z^3} + 2z \\ -\frac{1}{z^2} \\ 1 \end{bmatrix} = \begin{bmatrix} 17/4 \\ -1/4 \\ 1 \end{bmatrix}, \quad \ddot{\vec{r}} = \begin{bmatrix} -\frac{6}{z^4} + 2 \\ \frac{2}{z^3} \\ 0 \end{bmatrix} = \begin{bmatrix} 13/8 \\ 1/4 \\ 0 \end{bmatrix}, \quad \dddot{\vec{r}} = \begin{bmatrix} \frac{24}{z^5} \\ -\frac{6}{z^4} \\ 0 \end{bmatrix} = \begin{bmatrix} 3/4 \\ -3/8 \\ 0 \end{bmatrix}.$$

Sledi:

$$\|\dot{\vec{r}}\| = \frac{\sqrt{306}}{4}, \qquad \dot{\vec{r}} \times \ddot{\vec{r}} = \begin{bmatrix} -1/4 \\ 13/8 \\ 47/32 \end{bmatrix}, \qquad \|\dot{\vec{r}} \times \ddot{\vec{r}}\| = \frac{\sqrt{6897}}{32}, \qquad [\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}] = -\frac{51}{64}.$$

Torej je:

$$\kappa = \frac{2\sqrt{6897}}{306\sqrt{306}} \doteq 0{\cdot}0310, \qquad \omega = -\frac{816}{6897} \doteq 0{\cdot}118.$$

4. Iz standardne parametrizacije enotske sfere $\vec{r} = \begin{bmatrix} \cos\varphi\cos\theta \\ \cos\varphi\sin\theta \\ \sin\varphi \end{bmatrix}$ dobimo:

$$\vec{r}_\varphi = \begin{bmatrix} -\sin\varphi\cos\theta \\ -\sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix}, \quad \vec{r}_\theta = \begin{bmatrix} -\cos\varphi\sin\theta \\ \cos\varphi\cos\theta \\ 0 \end{bmatrix},$$

$$E = 1, \quad F = 0, \quad G = \cos^2\varphi, \quad EG - F^2 = \cos^2\varphi$$

in še:

$$u = \sqrt{x^2 + y^2 + (z-1)^2} = \sqrt{\cos^2\varphi + (\sin\varphi - 1)^2} = \sqrt{2(1-\sin\varphi)}$$

Sledi:

$$\iint_{x^2+y^2+z^2=1} u \, \mathrm{d}P = \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} \mathrm{d}\theta \sqrt{2(1-\sin\varphi)} \cos\varphi \, \mathrm{d}\varphi.$$

S substitucijo $t = 1 - \sin\varphi$ dobimo:

$$\iint_{x^2+y^2+z^2=1} u \, \mathrm{d}P = 2\pi\sqrt{2} \int_0^2 \sqrt{t} \, \mathrm{d}t = \frac{16\pi}{3}.$$

5. Uporabimo izrek o ostankih. Iz zapisa:

$$f(z) = \frac{e^z}{9\left(z - \frac{\pi i}{3}\right)\left(z + \frac{\pi i}{3}\right)}$$

je razvidno, da ima funkcija dva pola prve stopnje: enega v $\pi i/3$ in drugega v $-\pi i/3$. Toda le drugi leži znotraj krožnice, po kateri integriramo. Torej je:

$$\oint_K f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}\left(f, -\frac{\pi i}{3}\right).$$

Ker gre za pol prve stopnje, je:

$$\operatorname{Res}\left(f, -\frac{\pi i}{3}\right) = \lim_{z\to-\pi i/3}\left(z+\frac{\pi i}{3}\right) f(z) = \left.\frac{e^z}{z-\frac{\pi i}{3}}\right|_{z=-\pi i/3} = -\frac{\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}}{\frac{2\pi i}{3}} =$$
$$= \frac{3}{4\pi}\big(\sqrt{3}+i\big).$$

Sledi $\displaystyle\oint_K f(z)\,\mathrm{d}z = \frac{3}{2}\big(-1+i\sqrt{3}\big)$.