Rešitve izpita iz Matematike 2 z dne 29. 6. 2012

Praktična matematika

1. Če je $x = y = 0$, je očitno tudi $z = 0$. Po parcialnem odvajanju zveze dobimo:

$$\cos(xyz)\left(yz + xy\,\frac{\partial z}{\partial x}\right) - \frac{\partial z}{\partial x} = 0\,,$$
$$\cos(xyz)\left(xz + xy\,\frac{\partial z}{\partial y}\right) - \frac{\partial z}{\partial y} + 1 = 0\,.$$

Ko vstavimo $x = y = z = 0$, dobimo:

$$\frac{\partial z}{\partial x}(0,0) = 0\,, \qquad \frac{\partial z}{\partial y}(0,0) = 1\,.$$

Zdaj pa še enkrat parcialno odvajajmo po x. Dobimo:

$$-\sin(xyz)\left(yz + xy\,\frac{\partial z}{\partial x}\right)^2 + \cos(xyz)\left(2y\,\frac{\partial z}{\partial x} + xy\,\frac{\partial^2 z}{\partial x^2}\right) - \frac{\partial^2 z}{\partial x^2} = 0\,,$$

od koder po vstavitvi sledi še $\dfrac{\partial^2 z}{\partial x^2} = 0$.

2. Krivulja gre skozi izhodišče, ker je pri $t = 1$ (in samo tam) $x = y = z = 0$. Iz:

$$\dot{x} = \frac{1}{t} = 1\,, \qquad \dot{y} = e^{t-1} = 1\,, \qquad \dot{z} = 3t^2 - 2 = 1$$

sledi enačba tangente $x = y = z$. Nadalje iz:

$$\ddot{x} = -\frac{1}{t^2} = -1\,, \qquad \ddot{y} = e^{t-1} = 1\,, \qquad \ddot{z} = 6t = 6$$

dobimo, da v izhodišču velja:

$$\dot{\vec{r}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \ddot{\vec{r}} = \begin{bmatrix} -1 \\ 1 \\ 6 \end{bmatrix}, \quad \dot{\vec{r}} \times \ddot{\vec{r}} = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix},$$
$$\|\dot{\vec{r}}\| = \sqrt{3}\,, \qquad \|\dot{\vec{r}} \times \ddot{\vec{r}}\| = \sqrt{78}$$

in končno $\kappa = \dfrac{\|\dot{\vec{r}} \times \ddot{\vec{r}}\|}{\|\dot{\vec{r}}\|^3} = \dfrac{\sqrt{78}}{3\sqrt{3}} \doteq 1\text{·}700$.

3. Označimo iskani integral z J. S substitucijo $t = x^4$ dobimo:

$$J = \frac{1}{4}\int_0^1 t^{1/2}(1-t)^{1/2}\,\mathrm{d}t = \frac{\mathrm{B}\left(\frac{3}{2},\,\frac{3}{2}\right)}{4} = \frac{\left[\Gamma\left(\frac{3}{2}\right)\right]^2}{4\,\Gamma(3)} = \frac{\pi}{32}\,.$$

4. Označimo iskani integral z J. Najprej izračunamo:

$$EG - F^2 = 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + x^2 + y^2 .$$

Sledi:

$$J = \iint_{\substack{x,y \geq 0 \\ x^2+y^2 \leq 1}} xy \sqrt{1 + x^2 + y^2} \, \mathrm{d}x \, \mathrm{d}y .$$

Po uvedbi standardnih polarnih koordinat dobimo:

$$J = \int_O^{\pi/2} \cos\theta \sin\theta \, \mathrm{d}\theta \int_0^1 r\sqrt{1+r^2} \, \mathrm{d}r$$

Substituciji $s = \sin\theta$, $t = 1 + r^2$ nam dasta:

$$J = \frac{1}{2} \int_0^1 s \, \mathrm{d}s \int_1^2 \sqrt{t} \, \mathrm{d}t = \frac{2\sqrt{2} - 1}{6} .$$

5. Označimo:

$$f(z) = \frac{1}{(x^2 + 2z + 5)^2} = \frac{1}{(z + 1 + 2i)^2 (z + 1 - 2i)^2} .$$

Očitno je $\lim_{|z| \to \infty} z f(z) = 0$. Nadalje ima funkcija f dva pola druge stopnje: pri $-1 - 2i$ in $-1 + 2i$. Le slednji leži na zgornji polravnini, zato je:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = 2\pi i \operatorname{Res}(f, 1 + 2i) = \\
&= 2\pi i \left. \frac{\mathrm{d}}{\mathrm{d}z} \right|_{z=-1+2i} \frac{1}{(z + 1 + 2i)^2} = \\
&= - \left. \frac{4\pi i}{(z + 1 + 2i)^3} \right|_{-1+2i} = \\
&= \frac{\pi}{16} .
\end{aligned}$$