Rešitve izpita iz Matematike 2 z dne 27. 6. 2013

Praktična matematika

1. $f_x(x,y) = (-6y - y^2 + 6xy)\,e^{-x}$, $f_y(x,y) = (2y - 6x)\,e^{-x}$.
Stacionarni točki: $T_1(0,0)$, $T_2(2,6)$.
$f_{xx}(x,y) = (12y + y^2 + 6xy)\,e^{-x}$, $f_{xy}(x,y) = (-6 - 2y + 6x)\,e^{-x}$, $f_{yy}(x,y) = 2\,e^{-x}$.
V T_1 je $f_{xx} = 0$, $f_{xy} = -6$, $f_{yy} = 2$ in $K = -36$, zato tam ni ekstrema.
V T_2 pa je $f_{xx} = 180\,e^{-2}$, $f_{xy} = -6\,e^{-2}$, $f_{yy} = 2\,e^{-2}$ in $K = 324\,e^{-4}$, zato je tam lokalni minimum.

2. Krivuljo lahko parametriziramo z y:

$$x = \frac{e^y + e^{-y}}{2} = \operatorname{ch} y\,, \quad y = y\,, \quad z = \frac{e^y - e^{-y}}{2} = \operatorname{sh} y\,.$$

Po odvajanju in vstavljanju $y = 0$ dobimo:

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{e^y - e^{-y}}{2} = \operatorname{sh} y = 0\,, \quad \frac{\mathrm{d}y}{\mathrm{d}y} = 1\,, \quad \frac{\mathrm{d}z}{\mathrm{d}y} = \frac{e^y + e^{-y}}{2} = \operatorname{ch} y = 1\,.$$

Če torej s piko označimo odvod po y, je $\dot{\vec{r}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Po ponovnem odvajanju dobimo:

$$\frac{\mathrm{d}^2x}{\mathrm{d}y^2} = \frac{e^y + e^{-y}}{2} = \operatorname{ch} y = 1\,, \quad \frac{\mathrm{d}^2y}{\mathrm{d}y^2} = 0\,, \quad \frac{\mathrm{d}^2z}{\mathrm{d}y^2} = \frac{e^y - e^{-y}}{2} = \operatorname{sh} y = 0\,, \quad \ddot{\vec{r}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

in še:

$$\frac{\mathrm{d}^3x}{\mathrm{d}y^3} = \frac{e^y - e^{-y}}{2} = \operatorname{sh} y = 0\,, \quad \frac{\mathrm{d}^3y}{\mathrm{d}y^3} = 0\,, \quad \frac{\mathrm{d}^3z}{\mathrm{d}y^3} = \frac{e^y + e^{-y}}{2} = \operatorname{ch} y = 1\,, \quad \dddot{\vec{r}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Sledi:

$$\|\dot{\vec{r}}\| = \sqrt{2}\,, \quad \dot{\vec{r}} \times \ddot{\vec{r}} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \|\dot{\vec{r}} \times \ddot{\vec{r}}\| = \sqrt{2}\,, \quad \kappa = \frac{1}{2}$$

in še:

$$[\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}] = -1\,, \quad \omega = -\frac{1}{2}\,.$$

3. Označimo dano zgornjo polovico sfere z S_+. Iz standardne parametrizacije:

$$\vec{r} = \begin{bmatrix} \cos\varphi\cos\theta \\ \cos\varphi\sin\theta \\ \sin\varphi \end{bmatrix} \quad ; \qquad 0 \le \varphi \le \frac{\pi}{2}\,, \quad 0 \le \theta < 2\pi$$

dobimo:

$$\vec{r}_\varphi = \begin{bmatrix} -\sin\varphi\cos\theta \\ -\sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix} \quad , \quad \vec{r}_\theta = \begin{bmatrix} -\cos\varphi\sin\theta \\ \cos\varphi\cos\theta \\ 0 \end{bmatrix} \; ,$$

$$E = 1\,, \quad F = 0\,, \quad G = \cos^2\varphi\,, \quad EG - F^2 = \cos^2\varphi\,.$$

Sedaj lahko izračunamo maso:

$$m = \iint_{S_+} cz\,\mathrm{d}P = c\int_0^{2\pi}\int_0^{\pi/2} \sin\varphi\cos\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta = \pi c\,.$$

Zaradi simetrije ima težišče koordinati x in y enaki nič. Za izračun koordinate z potrebujemo še integral:

$$\iint_{S_+} cz^2\,\mathrm{d}P = c\int_0^{2\pi}\int_0^{\pi/2} \sin^2\varphi\cos\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta = \frac{2\pi c}{3}\,,$$

iz katerega dobimo $z^* = 2/3$. Z drugimi besedami, težišče je na dveh tretjinah višine.

4. Iz:

$$\operatorname{rot}\begin{bmatrix} 2\,x^a\,e^{y-z} \\ x^b\,e^{y-z} \\ -x^c\,e^{y-z} \end{bmatrix} = \begin{bmatrix} (x^b - x^c)\,e^{y-z} \\ (c\,x^{c-1} - 2\,x^a)\,e^{y-z} \\ (bx^{b-1} - 2\,x^a)\,e^{y-z} \end{bmatrix}$$

dobimo, da je polje potencialno natanko tedaj, ko je $b = c$, $a = c-1$, $c = 2$, $a = b-1$ in $b = 2$, torej natanko tedaj, ko je $a = 1$, $b = 2$ in $c = 2$. Iz nedoločenih integralov:

$$\int 2x\,e^{y-z}\,\mathrm{d}x = x^2\,e^{y-z} + C_1(y,z)\,,$$

$$\int x^2\,e^{y-z}\,\mathrm{d}y = x^2\,e^{y-z} + C_2(x,z)\,,$$

$$-\int x^2\,e^{y-z}\,\mathrm{d}z = x^2\,e^{y-z} + C_2(x,y)$$

dobimo potencial $u = x^2\,e^{y-z} + C$.

5. a) Pri $z = \pm\pi/2 + 2k\pi$, $k \in \mathbb{Z}$, so poli prve stopnje, pri $z = \pm\pi i/2$ pa sta pola druge stopnje.

b) Dana krožnica obkroži singularnosti v $\pi/2$ in $\pi i/2$. Ostanek v prvi singularnosti je enak:

$$\operatorname{Res}\left(f,\ \frac{\pi}{2}\right) = \lim_{z\to\pi/2}\left(z-\frac{\pi}{2}\right)f(z) = \lim_{z\to\pi/2}\frac{z-\frac{\pi}{2}}{\cos z(4z^2+\pi^2)} = \frac{1}{2\pi^2}\lim_{z\to\pi/2}\frac{z-\frac{\pi}{2}}{\cos z} = \\ = -\frac{1}{2\pi^2}\,,$$

ostanek v drugi singularnosti pa se izraža v obliki:

$$\operatorname{Res}\left(f,\ \frac{\pi i}{2}\right) = \lim_{z\to\pi i/2}\frac{\mathrm{d}}{\mathrm{d}z}\Big[(z-\pi i)^2 f(z)\Big]\,.$$

Iz zapisa:

$$f(z) = \frac{1}{\cos z(2z+\pi i)^2(2z-\pi i)^2}$$

dobimo:

$$\operatorname{Res}\left(f,\ \frac{\pi i}{2}\right) = \lim_{z\to\pi i/2}\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{\cos z\,(2z+\pi i)^2} = \\ = \frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=\pi i/2}\Big[\cos^{-1}z\,(2z+\pi i)^{-2}\Big] = \\ = \Big(\cos^{-2}z\,\sin z\,(2z+\pi i)^{-2} - 4\cos^{-1}z\,(2z+\pi i)^{-3}\Big)\bigg|_{z=\pi i/2}\,.$$

Izračunajmo:

$$\cos\frac{\pi i}{2} = \operatorname{ch}\frac{\pi}{2} = \frac{e^{\pi/2}+e^{-\pi/2}}{2}\,,\qquad \sin\frac{\pi i}{2} = i\operatorname{sh}\frac{\pi}{2} = \frac{e^{\pi/2}-e^{-\pi/2}}{2}\,i\,,$$

Sledi:

$$\operatorname{Res}\left(f,\ \frac{\pi i}{2}\right) = -\left(\frac{e^{\pi/2}-e^{-\pi/2}}{2\pi^2\big(e^{\pi/2}+e^{-\pi/2}\big)^2} + \frac{1}{\pi^3\big(e^{\pi/2}+e^{-\pi/2}\big)}\right)i$$

Po izreku o ostankih je iskani integral enak:

$$\oint_K f(z)\,\mathrm{d}z = 2\pi i\left[\operatorname{Res}\left(f,\ \frac{\pi}{2}\right) + \operatorname{Res}\left(f,\ \frac{\pi i}{2}\right)\right] = \\ = \frac{e^{\pi/2}-e^{-\pi/2}}{\pi\big(e^{\pi/2}+e^{-\pi/2}\big)^2} + \frac{2}{\pi^2\big(e^{\pi/2}+e^{-\pi/2}\big)} - \frac{i}{\pi}\,.$$