

# Rešitve kolokvija iz Matematike 2 z dne 10. 4. 2012

## Praktična matematika

- 1.** a) Najprej iz  $z = 0$  sledi  $u = 0$  ali pa  $v = 0$ . Če je  $u = 0$ , iz  $y = -2$  sledi  $v^2 = -2$ , kar ne more biti res. Če pa je  $v = 0$ , iz  $y = -2$  sledi  $u = -2$  in nato  $x = 4$ . Iskana točka je torej  $T(4, -2, 0)$ .

b) Iz prvih parcialnih odvodov:

$$\vec{r}_u = \begin{bmatrix} 2u \\ 1 \\ v \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{r}_v = \begin{bmatrix} 1 \\ 2v \\ u \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

dobimo koeficiente prve fundamentalne forme:

$$E = 17, \quad F = -4, \quad G = 5$$

in koeficient pri diferencialu ploščine je enak  $\sqrt{EG - F^2} = \sqrt{69}$ . Nadalje iz drugih parcialnih odvodov:

$$\vec{r}_{uu} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{r}_{uv} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{r}_{vv} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

dobimo še koeficiente druge fundamentalne forme:

$$L = -\frac{4}{\sqrt{69}}, \quad M = -\frac{1}{\sqrt{69}}, \quad N = -\frac{16}{\sqrt{69}}.$$

Gaussova ukrivljenost:  $K = \frac{63}{69^2} = \frac{7}{529}$ .

**2.** Velja:

$$F'(x) = \frac{\sin(x^2)}{x} - \frac{1}{2} \frac{\sin \frac{x^2}{2}}{\frac{x}{2}} + \int_{x/2}^x \cos(xy) dy = \frac{2 \left( \sin(x^2) - \sin \frac{x^2}{2} \right)}{x},$$

torej je  $F'(\sqrt{\pi}) = -\frac{2}{\sqrt{\pi}}$ .

**3.** Velja:

$$\begin{aligned} \int_0^\infty dx \int_0^{x+1} f(x, y) dy &= \iint_{\substack{x \geq 0 \\ 0 \leq y \leq x+1}} f(x, y) dx dy = \\ &= \int_0^1 dy \int_0^\infty f(x, y) dx + \int_1^\infty dy \int_{y-1}^\infty f(x, y) dx. \end{aligned}$$

4. Označimo iskani integral z  $J$ .

*Prvi način:* uvedemo običajne sferične koordinate:

$$x = r \cos \varphi \cos \theta, \quad y = r \cos \varphi \sin \theta, \quad z = r \sin \varphi; \quad J = r^2 \cos \varphi$$

in dobimo:

$$\begin{aligned} J &= \iiint_{\substack{0 \leq \theta < 2\pi \\ 0 \leq \varphi \leq \pi/2 \\ 0 \leq r \leq 2 \sin \varphi}} r^6 \cos^2 \theta \sin^2 \theta \cos^5 \varphi dr d\theta d\varphi = \\ &= \int_0^{2\pi} \cos^2 \theta \sin^2 \theta d\theta \int_0^{\pi/2} \int_0^{2 \sin \varphi} r^6 dr \cos^5 \varphi d\varphi = \\ &= 4 \int_0^{\pi/2} \cos^2 \theta \sin^2 \theta d\theta \cdot \frac{2^7}{7} \int_0^{\pi/2} \sin^7 \varphi \cos^5 \varphi d\varphi = \\ &= \frac{128}{7} B\left(\frac{3}{2}, \frac{3}{2}\right) B(4, 3) = \\ &= \frac{128}{7} \frac{\left[\Gamma\left(\frac{3}{2}\right)\right]^2}{\Gamma(3)} \frac{\Gamma(4) \Gamma(3)}{\Gamma(7)} = \\ &= \frac{128}{7} \cdot \frac{\pi}{8} \cdot \frac{1}{60} = \\ &= \frac{4\pi}{105}. \end{aligned}$$

*Drugi način:* uvedemo cilindrične koordinate:

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z; \quad J = r$$

in dobimo:

$$\begin{aligned}
J &= \iiint_{\substack{0 \leq \theta < 2\pi \\ r^2 + z^2 \leq 2z}} r^5 \cos^2 \theta \sin^2 \theta \, dr \, d\theta \, dz = \\
&= \iiint_{\substack{0 \leq \theta < 2\pi \\ 0 \leq r \leq 1 \\ 1 - \sqrt{1 - r^2} \leq z \leq 1 + \sqrt{1 - r^2}}} r^5 \cos^2 \theta \sin^2 \theta \, dr \, d\theta \, dz = \\
&= 2 \int_0^{2\pi} \cos^2 \theta \sin^2 \theta \, d\theta \int_0^1 r^5 \sqrt{1 - r^2} \, dr = \\
&= 4 \int_0^{\pi/2} \cos^2 \theta \sin^2 \theta \, d\theta \int_0^1 t^2 \sqrt{1 - t} \, dt = \\
&= 2 B\left(\frac{3}{2}, \frac{3}{2}\right) B\left(3, \frac{3}{2}\right) = \\
&= 2 \frac{\Gamma\left[\left(\frac{3}{2}\right)\right]^2}{\Gamma(3)} \frac{\Gamma(3) \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{9}{2}\right)} = \\
&= 2 \cdot \frac{\pi}{8} \cdot \frac{16}{105} = \\
&= \frac{4\pi}{105}.
\end{aligned}$$

*Tretji način:* uvedemo iste cilindrične koordinate kot prej, le da integriramo v drugačnem vrstnem redu:

$$\begin{aligned}
J &= \iiint_{\substack{0 \leq \theta < 2\pi \\ r^2 + z^2 \leq 2z}} r^5 \cos^2 \theta \sin^2 \theta \, dr \, d\theta \, dz = \\
&= \iiint_{\substack{0 \leq \theta < 2\pi \\ 0 \leq z \leq 2 \\ 0 \leq r \leq \sqrt{2z - z^2}}} r^5 \cos^2 \theta \sin^2 \theta \, dr \, d\theta \, dz = \\
&= 2 \int_0^{2\pi} \cos^2 \theta \sin^2 \theta \, d\theta \int_0^2 \int_0^{\sqrt{2z - z^2}} r^5 \, dr \, dz = \\
&= 8 \int_0^{\pi/2} \cos^2 \theta \sin^2 \theta \, d\theta \cdot \frac{1}{6} \int_0^2 (2z - z^2)^3 \, dz = \\
&= \frac{2}{3} B\left(\frac{3}{2}, \frac{3}{2}\right) \int_0^2 (8z^3 - 12z^4 + 6z^5 - z^6) \, dz = \\
&= \frac{2}{3} \frac{\left[\Gamma\left(\frac{3}{2}\right)\right]^2}{\Gamma(3)} \left(32 - \frac{384}{5} + 64 - \frac{128}{7}\right) = \\
&= \frac{\pi}{8} \cdot \frac{32}{35} = \\
&= \frac{4\pi}{105}.
\end{aligned}$$

*Četrti način:* uvedemo premaknjene sferične koordinate, in sicer tako, da je središče v točki  $(0, 0, 1)$ :

$$x = r \cos \varphi \cos \theta, \quad y = r \cos \varphi \sin \theta, \quad z = 1 + r \sin \varphi; \quad J = r^2 \cos \varphi.$$

Dobimo:

$$\begin{aligned} J &= \iiint_{\substack{0 \leq \theta < 2\pi \\ -\pi/2 \leq \varphi \leq \pi/2 \\ 0 \leq r \leq 1}} r^6 \cos^2 \theta \sin^2 \theta \cos^5 \varphi dr d\theta d\varphi = \\ &= \int_0^{2\pi} \cos^2 \theta \sin^2 \theta d\theta \int_{-\pi/2}^{\pi/2} \cos^5 \varphi d\varphi \int_0^1 r^6 dr = \\ &= 8 \int_0^{\pi/2} \cos^2 \theta \sin^2 \theta d\theta \int_0^{\pi/2} \cos^5 \varphi d\varphi \int_0^1 r^6 dr = \\ &= \frac{2}{7} B\left(\frac{3}{2}, \frac{3}{2}\right) B\left(3, \frac{1}{2}\right) = \\ &= \frac{2}{7} \frac{\left[\Gamma\left(\frac{3}{2}\right)\right]^2}{\Gamma(3)} \frac{\Gamma(3) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} = \\ &= \frac{2}{7} \cdot \frac{\pi}{8} \cdot \frac{16}{15} = \\ &= \frac{4\pi}{105}. \end{aligned}$$

5. Če z  $\rho$  označimo gostoto simpleksa, velja:

$$J_x = \rho \iiint_{\substack{x,y,z \geq 0 \\ x+y+z \leq 1}} (y^2 + z^2) dx dy dz.$$

Toda zaradi simetrije velja tudi kar:

$$\begin{aligned} J_x &= 2\rho \iiint_{\substack{x,y,z \geq 0 \\ x+y+z \leq 1}} z^2 dx dy dz = \\ &= 2\rho \int_0^1 \int_0^{1-z} \int_0^{1-y-z} dx dy z^2 dz = \\ &= 2\rho \int_0^1 \int_0^{1-z} (1-y-z) dy z^2 dz = \\ &= \rho \int_0^1 z^2 (1-z)^2 dz = \\ &= \rho B(3, 3) = \\ &= \rho \frac{[\Gamma(3)]^2}{\Gamma(6)} = \\ &= \frac{\rho}{30}. \end{aligned}$$