Rešitve kolokvija iz Matematike 2 z dne 16. 4. 2013

Praktična matematika

1. Velja:

$$F'(x) = \frac{1}{2}\sqrt{\left(1-\frac{x^2}{2}\right)\left(1+\frac{x}{2}\right)\left(1-\frac{x^2}{4}\right)} -$$
$$- \frac{1}{x}\sqrt{(1-x\ln x)(1+\ln x)\big(1-(\ln x)^2\big)} -$$
$$- \int_{\ln x}^{x/2} \frac{y\sqrt{(1+y)(1-y^2)}}{2\sqrt{(1-xy)}}\, \mathrm{d}y\,.$$

Sledi:

$$F'(1) = \frac{3}{8} - 1 - \frac{1}{2}\int_0^{1/2} y(1+y)\,\mathrm{d}y = -\frac{5}{8} - \frac{1}{16} - \frac{1}{48} = -\frac{17}{24}\,.$$

2. Označimo naš integral z I. S substitucijo:

$$u = xy, \quad v = \frac{y}{x}, \quad x = \sqrt{\frac{u}{v}}, \quad y = \sqrt{uv}, \quad J = \frac{1}{2v}$$

dobimo:

$$I = \frac{1}{2}\iint_{\substack{2<u<3\\1<v<2}} \mathrm{d}u\,\mathrm{d}v = \frac{1}{2}\,.$$

3. a) Označimo iskani integral z I_1. Z upoštevanjem sodosti, substitucijo $t = x^2$ in prevedbo na funkcijo beta dobimo:

$$I_1 = 2\int_0^\infty \frac{x^2}{(1+x^2)^2}\,\mathrm{d}x = \int_0^\infty \frac{\sqrt{t}}{(1+t)^2}\,\mathrm{d}t = \mathrm{B}\left(\frac{3}{2},\frac{1}{2}\right) = \frac{\Gamma\big(\frac{3}{2}\big)\,\Gamma\big(\frac{1}{2}\big)}{\Gamma(2)} =$$
$$= \frac{\frac{1}{2}\sqrt{\pi}\sqrt{\pi}}{1} = \frac{\pi}{2}\,.$$

b) Velja:

$$\int_0^\infty (x+2)^2\, e^{-x+1}\,\mathrm{d}x = e\int_0^\infty (x^2+4x+4)\,e^{-x}\,\mathrm{d}x = e\,(2!+4\cdot 1!+4\cdot 0!) = 10e\,.$$

4. Dani integral je dvojni integral po območju, določeno s pogojema:

$$1 < x < 2\,, \qquad x^2 < y < 2x\,,$$

ki sta ekvivalentna pogojem:

$$1 < y < 4\,, \qquad \frac{y}{2} < x < \sqrt{y}\,, \qquad 1 < x < 2\,.$$

Za $y \leq 2$ sta druga dva pogoja ekvivalentna pogoju $1 < x < \sqrt{y}$, za $y \geq 2$ pa pogoju $\frac{y}{2} < x < \sqrt{y}$. Integral z zamenjanim vrstnim redom je torej vsota integralov:

$$\int_1^2 \int_1^{\sqrt{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_2^4 \int_{y/2}^{\sqrt{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

5. Nalogo najlažje rešimo z vpeljavo sferičnih koordinat:

$$\begin{aligned} x &= r\cos\varphi\cos\theta \\ y &= r\cos\varphi\sin\theta \\ z &= r\sin\varphi \\ J &= r^2\cos\varphi\,, \end{aligned}$$

čeprav gre tudi s cilindričnimi. Masa:

$$m = c\iiint_{\substack{x^2+y^2+z^2<1\\ z>0}} z^2\,\mathrm{d}z = c\iiint_{\substack{0\leq r<1\\ 0\leq\theta<2\pi\\ 0\leq\varphi\leq\pi/2}} r^4\sin^2\varphi\cos\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta = \frac{2c\pi}{15}\,.$$

Zaradi simetrije je težišče v točki $T^*(0,0,z^*)$, kjer je:

$$z^* = \frac{c}{m}\iiint_{\substack{x^2+y^2+z^2<1\\ z>0}} z^3\,\mathrm{d}z = \frac{c}{m}\iiint_{\substack{0\leq r<1\\ 0\leq\theta<2\pi\\ 0\leq\varphi\leq\pi/2}} r^5\sin^3\varphi\cos\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta = \frac{c\pi}{12m} = \frac{5}{8}\,.$$

Vztrajnostni moment okoli osi z pa je:

$$\begin{aligned} J_z &= c\iiint_{\substack{x^2+y^2+z^2<1\\ z>0}} (x^2+y^2)z^2\,\mathrm{d}z = c\iiint_{\substack{0\leq r<1\\ 0\leq\theta<2\pi\\ 0\leq\varphi\leq\pi/2}} r^6\sin^2\varphi\cos^3\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta = \\ &= c\iiint_{\substack{0\leq r<1\\ 0\leq\theta<2\pi\\ 0\leq\varphi\leq\pi/2}} r^6(\sin^2\varphi - \sin^4\varphi)\cos\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta = \frac{4c\pi}{105}\,. \end{aligned}$$