Rešitve kolokvija iz Matematike 2 z dne 6. 6. 2011

Praktična matematika

1. Iz:

$$\operatorname{rot}\vec{R} = \begin{bmatrix} (a+2)(x+y)^a \\ -(a+2)(x+y)^a \\ 0 \end{bmatrix}$$

razberemo, da je polje potencialno natanko tedaj, ko je $a = -2$. Njegov potencial je polje $u(x,y,z) = \dfrac{z}{x+y}$.

2. Najprej izračunamo $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x$ in $\dfrac{\mathrm{d}z}{\mathrm{d}x} = 2x^2$. Če dolžinsko gostoto označimo z ρ, velja:

$$\begin{aligned} m &= \int_{-1}^{1} \rho\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \\ &= \rho\int_{-1}^{1} \sqrt{1+4x^2+4x^4}\,\mathrm{d}x = \\ &= 2\rho\int_{0}^{1} (1+2x^2)\,\mathrm{d}x = \\ &= \frac{10}{3}\,\rho\,. \end{aligned}$$

Težišče je točka s koordinatami (x^*, y^*, z^*), kjer je:

$$\begin{aligned} x^* &= \frac{1}{m}\int_{-1}^{1}\int \rho x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \frac{3}{10}\int_{-1}^{1} x(1+2x^2)\,\mathrm{d}x = 0\,, \\ y^* &= \frac{1}{m}\int_{-1}^{1}\int \rho y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \frac{3}{10}\int_{-1}^{1} x^2(1+2x^2)\,\mathrm{d}x = \frac{11}{25}\,, \\ z^* &= \frac{1}{m}\int_{-1}^{1}\int \rho z\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \frac{3}{10}\int_{-1}^{1} \frac{2x^3}{3}(1+2x^2)\,\mathrm{d}x = 0\,. \end{aligned}$$

3. Plašč valja najprej parametriziramo:

$$x = \cos\theta\,, \quad y = \sin\theta\,, \quad z = z\,; \qquad 0 \le \theta < 2\pi\,, \quad z \in \mathbb{R}\,.$$

in v izbranih koordinatah zapišemo vektorsko polje, ki ga integriramo:

$$\vec{R} = \frac{1}{x^2+y^2+z^2}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1+z^2}\begin{bmatrix} \cos\theta \\ \sin\theta \\ z \end{bmatrix}$$

Nato izračunamo:

$$\vec{r}_\theta = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix}, \quad \vec{r}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{r}_\theta \times \vec{r}_z = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}.$$

Ker zgornji vektorski produkt vedno kaže iz valja, je pretok enak:

$$\Phi = \iint_{\substack{0\le\theta<2\pi \\ z\in\mathbb{R}}} \frac{1}{1+z^2} \left\langle \begin{bmatrix} \cos\theta \\ \sin\theta \\ z \end{bmatrix}, \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} \right\rangle \mathrm{d}\theta\,\mathrm{d}z = \int_0^{2\pi} \mathrm{d}\theta \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{1+z^2} = 2\pi^2 .$$

4. Če pišemo $f(z) = g(w)$, kjer je $z = -2 + w$, velja:

$$g(w) = \frac{1}{w(w-3)} = -\frac{1}{3w\left(1-\frac{w}{3}\right)} = -\frac{1}{3w} - \frac{1}{9} - \frac{w}{27} - \frac{w^2}{81} - \cdots .$$

Sledi:

$$\begin{aligned} f(z) &= -\frac{1}{3(z+2)} - \frac{1}{9} - \frac{z+2}{27} - \frac{(z+2)^2}{81} - \cdots = \\ &= -\sum_{n=-1}^{\infty} \frac{1}{3^{n+2}} (z+2)^n . \end{aligned}$$

5. Substitucija $z = e^{ix}$, $\cos x = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$, $\mathrm{d}x = \dfrac{\mathrm{d}z}{iz}$, nam da:

$$I := \int_0^{2\pi} \frac{\mathrm{d}x}{3 + 2\cos x} = \frac{1}{i} \oint_K f(z)\,\mathrm{d}z ,$$

kjer je K pozitivno orientirana enotska krožnica, funkcija f pa je definirana po predpisu:

$$f(z) = \frac{1}{z^2 + 3z + 1} = \frac{1}{\left(z + \frac{3}{2} + \frac{1}{2}\sqrt{5}\right)\left(z + \frac{3}{2} - \frac{1}{2}\sqrt{5}\right)} .$$

Funkcija f ima dve singularnosti, $-\frac{3}{2} + \frac{1}{2}\sqrt{5}$ in $-\frac{3}{2} - \frac{1}{2}\sqrt{5}$. Le prva leži znotraj enotskega kroga, zato velja:

$$I = 2\pi \operatorname{Res}\left(f, -\tfrac{3}{2} + \tfrac{1}{2}\sqrt{5}\right) .$$

Ker gre za pol prve stopnje, je:

$$\begin{aligned} \operatorname{Res}\left(f, -\tfrac{3}{2} + \tfrac{1}{2}\sqrt{5}\right) &= \lim_{z\to -\frac{3}{2}+\frac{1}{2}\sqrt{5}} \left(z + \tfrac{3}{2} - \tfrac{1}{2}\sqrt{5}\right) f(z) = \\ &= \left.\frac{1}{\left(z + \frac{3}{2} + \frac{1}{2}\sqrt{5}\right)}\right|_{z=-\frac{3}{2}+\frac{1}{2}\sqrt{5}} = \\ &= \frac{1}{\sqrt{5}} . \end{aligned}$$

Sledi $I = \dfrac{2\pi\sqrt{5}}{5}$.