Rešitve 2. kolokvija iz Analize 2 (2005/6)

1. Za naslednji številski vrsti ugotovi, ali konvergirata:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n \cdot 2^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n^3}{2^n}$

Svoja odgovora dobro utemelji.

(a) $$D_n = \frac{a_{n+1}}{a_n} = \frac{3^{n+1}\, n \cdot 2^n}{(n+1)2^{n+1} \cdot 3^n} = \frac{3n}{2(n+1)}$$

$\lim_{n\to\infty} D_n = \frac{3}{2} > 1 \Rightarrow$ vrsta divergira

(b) Vrsta alternira

$$\lim_{n\to\infty} \frac{n^3}{2^n} = \lim_{n\to\infty} \frac{3n^2}{2^n \ln 2} = \lim_{n\to\infty} \frac{6n}{2^n(\ln 2)^2} = \lim_{n\to\infty} \frac{6}{2^n(\ln 2)^3} = 0$$

$$\frac{(n+1)^3}{2^{n+1}} < \frac{n^3}{2^n} \Leftrightarrow \left(\frac{n+1}{n}\right)^3 < 2 \Leftrightarrow 1 + \frac{1}{n} < \sqrt[3]{2}$$

to je res za dovolj velike n

Njeni členi monotono padajo proti 0

$\Rightarrow$ konvergira

(b) alternativna rešitev

Vrsta absolutno konvergira, ker

$\sum \frac{n^3}{2^n}$ konvergira: $\quad C_n = \sqrt[n]{\frac{n^3}{2^n}} \xrightarrow{n\to\infty} \frac{1}{2} < 1$

zato tudi (pogojno) konvergira

2. (a) V Taylorjevo vrsto okrog točke 0 razvij funkcijo

$$f(x) = \frac{x}{1+x^2}.$$

(*Nasvet:* Pomagaj si z geometrijsko vrsto.)

(b) S pomočjo prejšnje točke poišči vrednost $f^{(7)}(0)$. (*Nasvet:* Primerjaj istoležne koeficiente pri potencah x.)

(a) $\frac{x}{1+x^2} = x \cdot \frac{1}{1+x^2} = x\left(1 - x^2 + x^4 - x^6 + x^8 - \dots\right) =$

$= x - x^3 + x^5 - x^7 + x^9 - \dots$

(b) koef. pri x^7 je:

$$\frac{f^{(7)}(0)}{7!} = -1 \Rightarrow f^{(7)}(0) = -7!$$

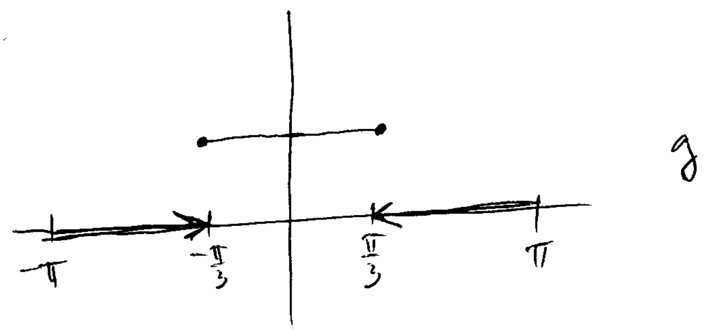
3. (a) Funkcija $f\colon\ [0,\pi) \to \mathbb{R}$ je definirana s pravilom

$$f(x) = \begin{cases} 1, & \text{če je } 0 \leqslant x \leq \frac{\pi}{3}, \\ 0, & \text{če je } \frac{\pi}{3} < x < \pi. \end{cases}$$

Funkcijo f razširi do sode funkcije $g\colon\ (-\pi,\pi) \to \mathbb{R}$ in nariši njen graf.

(b) Funkcijo g razvij po samih kosinusih na intervalu $(-\pi,\pi)$.

(a)



(b) $$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

Ker je soda, $b_n = 0$.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\,dx = \frac{1}{2\pi} \int_{-\pi/3}^{\pi/3} 1\,dx = \frac{1}{2\pi} \cdot \frac{2\pi}{3} = \frac{1}{3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos nx\,dx = \frac{1}{\pi} \int_{-\pi/3}^{\pi/3} \cos nx\,dx = \frac{1}{\pi} \left.\frac{\sin nx}{n}\right|_{-\pi/3}^{\pi/3} =$$

$$= \frac{1}{\pi n} \cdot 2 \cdot \sin \frac{n\pi}{3}$$

$$g(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{\pi n} \sin \frac{n\pi}{3} \cos nx$$

4. V prostoru je dana ravnina Σ: $3x - 2z = 0$ in točki $A(1,1,1)$, $B(2,3,4)$. Poišči takšno točko T na ravnini Σ, da bo vsota kvadratov oddaljenosti točke T od točk A in B:

$$d(T,A)^2 + d(T,B)^2$$

najmanjša možna. Preveri, da si v točki T res dobil minimum.

$$f(x,y,z;\lambda) = (x-1)^2 + (y-1)^2 + (z-1)^2 + (x-2)^2 + (y-3)^2 + (z-4)^2 - \lambda(3x-2z)$$

$$0 = f_x = 2(x-1) + 2(x-2) - 3\lambda$$
$$0 = f_y = 2(y-1) + 2(y-3)$$
$$0 = f_z = 2(z-1) + 2(z-4) + 2\lambda$$
$$0 = f_\lambda = -3x + 2z$$

$$4y = 8 \Rightarrow y = 2$$
$$z = \frac{3x}{2}$$

$$4x - 6 - 3\lambda = 0$$
$$4 \cdot \frac{3x}{2} - 10 + 2\lambda = 0$$

$$4x - 3\lambda = 6 \qquad 12x - 9\lambda = 18$$
$$6x + 2\lambda = 10 \qquad 12x + 4\lambda = 20$$

$$-13\lambda = -2$$
$$\lambda = \frac{2}{13}$$
$$x = \frac{6 + \frac{6}{13}}{4} = \frac{84}{13 \cdot 4} = \frac{21}{13}, \quad z = \frac{3}{2} \cdot \frac{21}{13} = \frac{63}{26}$$

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

gl. minorji $4, 4^2, 4^3 > 0$

$4 > 0$

MIN.