1. sklop dodatnih vaj iz Analize 2

(1) Izračunaj nedoločene integrale:

(a) $\displaystyle\int 2x \operatorname{arc\,tg} x \, dx$,

(b) $\displaystyle\int \frac{\arcsin\left(\frac{1}{x}\right)}{x^2} \, dx$,

(c) $\displaystyle\int \frac{1}{(\operatorname{arc\,tg} x)(1+x^2)} \, dx$,

(d) $\displaystyle\int e^x \left(\operatorname{arc\,sin} x + \frac{1}{\sqrt{1-x^2}}\right) dx$,

(e) $\displaystyle\int (\cos x - \sin x)\sqrt{3 + \cos x + \sin x} \, dx$.

Rešitev:

(a) $I = -x + (x^2+1) \operatorname{arc\,tg} x + C$,

(b) $I = -\sqrt{1 - \dfrac{1}{x^2}} - \dfrac{\operatorname{arc\,sin}\left(\frac{1}{x}\right)}{x} + C$,

(c) $I = \ln|\operatorname{arc\,tg} x| + C$,

(d) $I = e^x \operatorname{arc\,sin} x + C$,

(e) $I = \dfrac{2}{3}(3 + \cos x + \sin x)^{\frac{3}{2}} + C$.

(2) Izračunaj nedoločene integrale:

(a) $\displaystyle\int \frac{1}{(x+1)^2(x+2)} \, dx$,

(b) $\displaystyle\int \frac{20x}{(x+1)(x^2+9)} \, dx$,

(c) $\displaystyle\int \frac{54}{(x^2+8x+25)^2} \, dx$.

Rešitev:

(a) $I = -\dfrac{1}{x+1} - \ln|x+1| + \ln|x+2| + C$,

(b) $I = -2\ln|x+1| + \ln|x^2+9| + 6 \operatorname{arc\,tg}\left(\dfrac{x}{3}\right) + C$,

(c) $I = \dfrac{12+3x}{x^2+8x+25} + \operatorname{arc\,tg}\left(\dfrac{x+4}{3}\right) + C$.

(3) Izračunaj nedoločene integrale:

(a) $\displaystyle\int \frac{1}{\sin x \cos x}\, dx$,

(b) $\displaystyle\int \frac{1}{1+2\sin^2 x}\, dx$,

(c) $\displaystyle\int e^x \sqrt{1+e^x}\, dx$,

(d) $\displaystyle\int \frac{1-\sqrt{x+2}}{1+\sqrt{x+2}}\, dx$,

(e) $\displaystyle\int \frac{x-1}{\sqrt{x^2-4}}\, dx$,

(f) $\displaystyle\int \sqrt{1+x^2}\, dx$.

Rešitev:

(a) $I = \ln|\operatorname{tg} x| + C$,

(b) $I = \dfrac{\sqrt{3}}{3} \operatorname{arc\,tg}\left(\sqrt{3}\operatorname{tg} x\right) + C$,

(c) $I = \dfrac{2}{3}(1+e^x)^{\frac{3}{2}} + C$,

(d) $I = 3 - x + 4\sqrt{x+2} - 4\ln|1+\sqrt{x+2}| + C$,

(e) $I = \sqrt{x^2-4} - \ln|x+\sqrt{x^2-4}| + C$,

(f) $I = \dfrac{1}{2}\left(x\sqrt{1+x^2} + \ln(x+\sqrt{1+x^2})\right) + C$.

(4) Izračunaj določene integrale:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{2+\cos x}\, dx$,

(b) $\displaystyle\int_0^1 x^2 \operatorname{arc\,tg}\left(x^3\right)\, dx$,

(c) $\displaystyle\int_0^1 \frac{1}{(x+1)(x^2+1)}\, dx$.

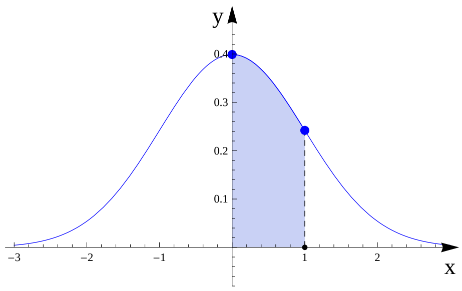
Rešitev:

(a) $I = \dfrac{\pi}{3\sqrt{3}}$,

(b) $I = \dfrac{\pi}{12} - \dfrac{1}{6}\ln 2$,

(c) $I = \dfrac{\pi}{8} + \dfrac{1}{4}\ln 2$.

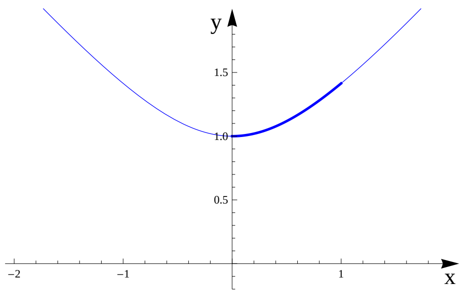
(5) S trapezno metodo izračunaj ploščino lika pod grafom funkcije $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ na intervalu $[0, 1]$, tako da bo napaka manjša od 0.01.



Rešitev: $S = 0.34$ dobimo z uporabo trapezne metode pri $n = 4$.

Zaokrožena natančna vrednost je $S = 0.3413$.

(6) Z uporabo trapezne metode izračunaj dolžino hiperbole $f(x) = \sqrt{1 + x^2}$ na intervalu $[0, 1]$, tako da bo napaka manjša od 0.01.



Rešitev: $l = 1.10$ dobimo z uporabo trapezne metode pri $n = 8$.

Zaokrožena natančna vrednost je $l = 1.09969$.

(7) Izračunaj izlimitirane integrale:

(a) $\displaystyle\int_1^{\infty} \frac{\operatorname{arc\,tg} x}{x^2}\, dx$,

(b) $\displaystyle\int_0^{\infty} \frac{1}{x^2 + 2x + 2}\, dx$,

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \operatorname{tg} x\, dx$.

Rešitev:

(a) $I = \dfrac{1}{4}(\pi + \ln 4)$,

(b) $I = \dfrac{\pi}{4}$,

(c) integral divergira.

(8) Ugotovi, ali izlimitirani integrali konvergirajo ali divergirajo:

(a) $\displaystyle\int_1^2 \frac{1}{\sqrt{8-x^3}}\,dx$,

(b) $\displaystyle\int_1^\infty \frac{\ln x}{x^2\sqrt{x-1}}\,dx$,

(c) $\displaystyle\int_0^\infty \frac{1}{\sqrt{x}+\sqrt[4]{x}+x^2}\,dx$,

(d) $\displaystyle\int_1^\infty \frac{\ln x}{(x-1)^2}\,dx$.

Rešitev:

(a) integral konvergira,

(b) integral konvergira,

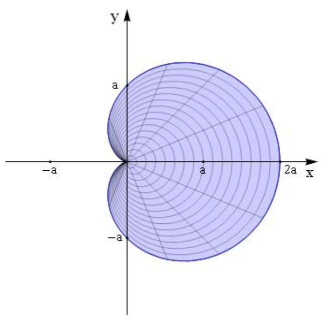
(c) integral konvergira,

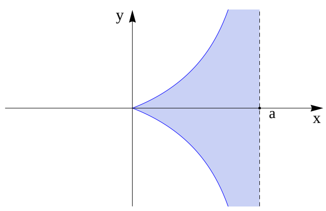
(d) integral divergira.

(9) Izračunaj ploščini naslednjih likov:

(a) lika, ki ga omejuje kardioida $r = a(1+\cos\phi)$, za $a > 0$,

(b) lika, ki ga omejujeta strofoida $y^2(a-x) = x^2(a+x)$ in premica $x = a$, za $a > 0$.





Rešitev:

(a) $S = \dfrac{3\pi a^2}{2}$,

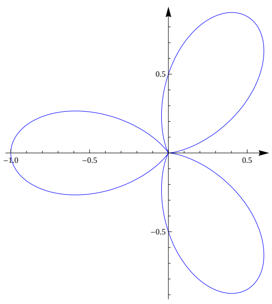
(b) $S = a^2\left(\dfrac{\pi}{2}+2\right)$.

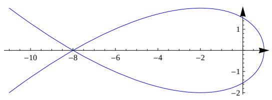
(10) Izračunaj dolžine naslednjih krivulj:

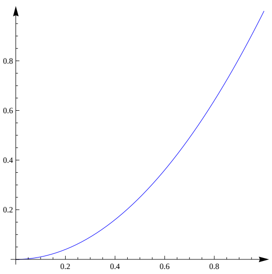
(a) krivulje v polarnih koordinatah $r(\phi) = \sin^2(3\phi/2)$ za $\phi \in [0, 2\pi]$,

(b) parametrično podane krivulje $\vec{r}(t) = \left(1-3t^2, 3t-t^3\right)$ za $t \in [-2, 2]$,

(c) grafa funkcije $f(x) = x^2$ za $x \in [0, 1]$.







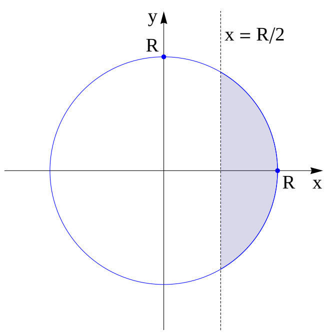
Rešitev:

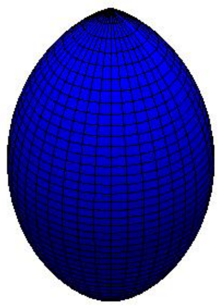
(a) $l = 6 + \dfrac{\sqrt{2}}{2}\ln(3 + 2\sqrt{2})$,

(b) $l = 28$,

(c) $l = \dfrac{\sqrt{5}}{2} + \dfrac{1}{4}\ln(2 + \sqrt{5})$.

(11) Dan je krožni odsek $K = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq R^2, x \geq \frac{R}{2}\}$, kjer je R pozitivno realno število. Izračunaj volumen in površino telesa, ki ga dobimo, če dani krožni odsek zavrtimo okoli osi $x = \frac{R}{2}$.

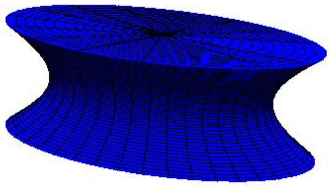




Rešitev: $V = \pi R^3 \left(\frac{3\sqrt{3}}{4} - \frac{\pi}{3}\right)$, $P = 2\pi R^2 \left(\sqrt{3} - \frac{\pi}{3}\right)$.

(12) Izračunaj volumen hiperboloida, danega z neenačbami

$$T = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + 4y^2 - 4z^2 \leq 4, \ -1 \leq z \leq 1\}.$$

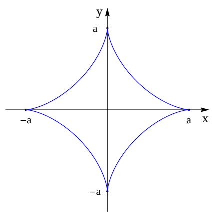


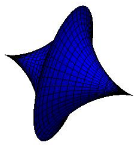
Rešitev: $V = \frac{16\pi}{3}$.

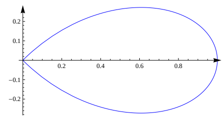
(13) Izračunaj površini vrtenin, ki ju dobimo z vrtenjem:

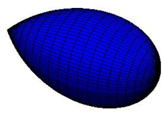
(a) astroide $x^{2/3} + y^{2/3} = a^{2/3}$, za $a > 0$, okoli abscisne osi,

(b) lemniskate $r = a\sqrt{\cos 2\phi}$, za $a > 0$, okoli abscisne osi.









Rešitev: (a) $P = \frac{12\pi a^2}{5}$, (b) $P = 2\pi a^2 \left(2 - \sqrt{2}\right)$.

(14) Dana je krivulja

$$\vec{r}(t) = \left(\sqrt{1+t^2}, t, \ln\left(t + \sqrt{1+t^2}\right)\right).$$

Izračunaj ukrivljenost, torzijo in Frenetovo ogrodje v točki $\vec{r}(0)$.

Rešitev: $\kappa = \tau = \frac{1}{2}$, $\vec{T}(0) = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\vec{N}(0) = (1,0,0)$, $\vec{B}(0) = \left(0, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

(15) Krivulja je podana parametrično s predpisom $\vec{r}(t) = (t^2 + t, t + 1, t^3 - 1)$.

(a) Izračunaj ukrivljenost in torzijo krivulje v točki $T(0, 1, -1)$.

(b) Izračunaj enačbo pritisnjene ravnine v točki $T(0, 0, -2)$.

Rešitev:

(a) $\kappa = \dfrac{\sqrt{2}}{2}$, $\tau = -3$,

(b) $\Pi:\ 3x + z = -2$.