b) <u>Metoda max. zanesljivosti</u> (<u>največjega verjetja</u>)

Naj bo gostota za X odvisna od ξ: $p(x;\xi)$. <u>Funkcija zanesljivosti</u> (likelihood function) je $L(x_1,\dots,x_n;\xi) = p(x_1;\xi)\cdots\cdots p(x_n;\xi)$

Pri danih $x_1,\dots,x_n$ določimo tak ξ_{max}, da ima L tam maksimum. Ta vrednost je odvisna od $x_1,\dots,x_n$, torej $\xi_{max} = \varphi(x_1,\dots x_n)$

Tako dobimo cenilko za ξ:

$$C = \varphi(X_1, \dots, X_n)$$

Zgledi: 1) $X \sim Exp(\lambda)$: $p(x;\lambda) = \lambda \cdot e^{-\lambda x}$ za $x > 0$, sicer je $p(x;\lambda) = 0$.

$$L(x_1, \dots, x_n; \lambda) = \lambda e^{-\lambda x_1} \cdot \ldots \cdot \lambda \cdot e^{-\lambda x_n} = \lambda^n \cdot e^{-\lambda(x_1 + \dots + x_n)}$$

$$\ln L = n \cdot \ln \lambda - \lambda(x_1 + \dots + x_n)$$

$$\frac{\partial}{\partial \lambda}(\ln L) = \frac{n}{\lambda} - (x_1 + \dots + x_n) = 0 \Rightarrow \lambda = \frac{n}{x_1 + \dots + x_n} = \frac{1}{\bar{x}}$$

$$\frac{\partial^2}{\partial \lambda^2}(\ln L) = -\frac{n}{\lambda^2} < 0$$, zato imamo max.

Torej dobimo cenilko $C = \frac{1}{\overline{X}}$ za λ.

Isto cenilko dobimo z metodo momentov:

$$z_1 = E(X) = \int_0^\infty x \cdot \lambda \cdot e^{-\lambda x}\, dx \underset{t=\lambda x}{=} \frac{1}{\lambda}\int_0^\infty t \cdot e^{-t}\, dt$$

$$= \frac{1}{\lambda}\Gamma(2) = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{z_1} \Rightarrow C = \frac{1}{Z_1} = \frac{1}{\overline{X}}.$$

2) več parametrov: $X \sim N(\mu, \sigma)$

$$p(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$L(x_1, \ldots, x_n; \mu, \sigma) = \frac{1}{\sigma^n \cdot (2\pi)^{n/2}}\, e^{-\frac{1}{2\sigma^2}\left((x_1-\mu)^2 + \ldots + (x_n-\mu)^2\right)}$$

$$\ln L = -n\ln\sigma - \frac{n}{2}\ln(2\pi) - \frac{1}{2\sigma^2}\left((x_1-\mu)^2+\dots+(x_n-\mu)^2\right)$$

$$\frac{\partial}{\partial\mu}(\ln L) = +\frac{1}{2\sigma^2}\left(2(x_1-\mu)+\dots+2(x_n-\mu)\right) = 0$$

$$x_1+\dots+x_n - n\cdot\mu = 0 \Rightarrow \mu = \bar{x}$$

$$\frac{\partial}{\partial\sigma}(\ln L) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\left((x_1-\mu)^2+\dots+(x_n-\mu)^2\right) = 0$$

$$\Rightarrow \sigma^2 = \frac{1}{n}\left((x_1-\mu)^2+\dots+(x_n-\mu)^2\right) \underset{\mu=\bar{x}}{=} s_0^2$$

$\Rightarrow$ cenilka za μ je $\overline{X}$, za σ je S_0

3) $X \sim Ber(p)$: $X: \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$ Cenilka za p?

$$P(X=x) = p^{x} \cdot 2^{1-x} \quad \text{za } x \in \{0, 1\}$$

$$L(x_1, \dots, x_n; p) = p^{x_1} \cdot (1-p)^{1-x_1} \cdot \ldots \cdot p^{x_n} \cdot (1-p)^{1-x_n} =$$

$$= p^{x_1+\dots+x_n} \cdot (1-p)^{n-(x_1+\dots+x_n)} = p^{x} \cdot (1-p)^{n-x}, \text{ kjer}$$

$x = x_1 + \dots + x_n \in \{0, 1, 2, \dots, n\}$

$$\ln L = x \cdot \ln p + (n-x) \cdot \ln(1-p)$$

$$\frac{\partial}{\partial p}(\ln L) = \frac{x}{p} - \frac{n-x}{1-p} = 0 \Rightarrow$$

$$0 = x(1-p) - p(n-x) = x - xp - pn + px \Rightarrow p = \frac{x}{n} = \bar{x}$$

Cenilka za p je $\overline{X}$

4. Intervalsko ocenjevanje parametrov

Naj bo porazdelitev za X odvisna od ξ

Naj bo $(X_1, X_2, \ldots, X_n)$ vzorec.

Interval $[A,B]$ (odvisen od vzorca in ne od ξ) je **interval zaupanja za ξ pri stopnji tveganja** $\alpha \in (0,1)$, če je

$$P(\xi \in [A,B]) = 1-\alpha \quad \text{oz.} \quad P(\xi \notin [A,B]) = \alpha$$

A in B sta vzorčni statistiki, $1-\alpha$ je **stopnja zaupanja**

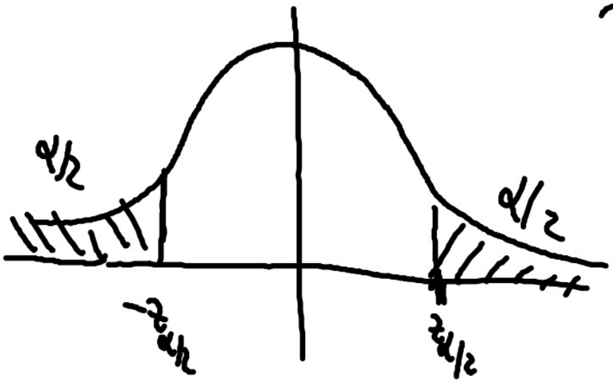
Običajno $\alpha = 0.05$ ali $\alpha = 0.01$

Primeri 1) $X \sim N(\mu, \sigma)$, σ poznamo, μ ocenjujemo

$$\bar{X} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \Rightarrow Z = \frac{\bar{X} - \mu}{\sigma}\sqrt{n} \sim N(0,1)$$

Pri izbranem $\alpha \in (0,1)$ določimo $z_{\alpha/2} > 0$, da je $P(Z > z_{\alpha/2}) = \frac{\alpha}{2}$ oz.

$P(|Z| > z_{\alpha/2}) = \alpha$ oz. $P(|Z| \leq z_{\alpha/2}) = 1 - \alpha$



Pogoj $|Z| \leq z_{\alpha/2}$ pomeni:

$$|\bar{X} - \mu| \leq z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \iff$$

$$\bar{X} - z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

Interval zaupanja $[A, B]$ je $A = \overline{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$,
$B = \overline{X} + z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$

Konkreten zgled: Imejmo vzorec vel. 36, za katerega naračunamo $\overline{x} = 2{.}6$ in $s = 0{.}3$.
Predpostavimo $\sigma = s = 0{.}3$. Vzemimo $\alpha = 0{.}05$.
Potem je $z_{\alpha/2} = 1{.}96$. Tedaj je vzorčna vrednost za A je $\overline{x} - z_{\alpha/2} \cdot \frac{s}{\sqrt{n}} = 2{.}6 - 1{.}96 \cdot \frac{0{.}3}{6} = 2{.}5$,
za B pa $2{.}6 + 1{.}96 \cdot \frac{0{.}3}{6} = 2{.}7$

$P(\mu \in [2{.}5, 2{.}7]) = 0{.}95$

2) $X \sim N(\mu, \sigma)$, σ ne poznamo, μ pa ocenjujemo. Studentova statistika

$$T = \frac{\overline{X} - \mu}{S} \cdot \sqrt{n} \sim Student(n-1)$$

Pri danem stopnji tveganja $\alpha \in (0,1)$ določimo $t_{\alpha/2} > 0$, da je $P(T > t_{\alpha/2}) = \frac{\alpha}{2}$ oz.

$P(|T| \leq t_{\alpha/2}) = 1 - \alpha$. Kot prej dobimo:

$$A = \overline{X} - t_{\alpha/2} \cdot \frac{S}{\sqrt{n}}, \quad B = \overline{X} + t_{\alpha/2} \cdot \frac{S}{\sqrt{n}}$$