6. FORMULA INVERZIJE IN BINOMSKA INVERZIJA

- $\mathcal{M}$ ... neskončne kompleksne matrike; vrstice, stolpci indeksirani z $\mathbb{N}_0$

Če sta $A, B \in \mathcal{M}$, je $A \cdot B$ smiseln, če ima vsak produkt $A_i \cdot B_j$ končno mnogo neničelnih elt. To bo vedno res, če je A spodnje-trikotna ali B zgornje trikotna.

- $\mathcal{M}^\ell$ ... spodnje trikotne matrike, $\mathcal{M}^\ell \subseteq \mathcal{M}$: $a_{ij} = 0$ za $i < j$
- $\mathcal{M}^u$ ... zgornje trikotne matrike, $\mathcal{M}^u \subseteq \mathcal{M}$: $b_{ij} = 0$ za $i > j$.
- $A \in \mathcal{M}^\ell$ ima inverz $\Leftrightarrow$ $a_{ii} \neq 0 \ \forall i$. Tedaj je tudi $A^{-1} \in \mathcal{M}^\ell$.
- Analogno za $\mathcal{M}^u$.

Spomnimo se, če sta (p_n) in (q_n) polinomski zaporedji, tedaj imamo:

$$p_n(x) = \sum_{k=0}^{n} a_{n,k} q_k(x) \quad \text{in} \quad q_n(x) = \sum_{k=0}^{n} b_{n,k} p_k(x),$$

kjer so $a_{n,k}$ in $b_{n,k}$ povezovalni koeficienti. Ker sta $(p_n(x))$ in $(q_n(x))$ bazi prostora $\mathbb{C}[x]$, sta matriki

$$A = (a_{n,k}) \quad \text{in} \quad B = (b_{n,k})$$

inverzni:

$$A \cdot B = I$$

Naj bosta sedaj

$$u = (u_0, u_1, \dots) \quad \text{in} \quad v = (v_0, v_1, \dots) \text{ zaporedji kompleksnih števil.}$$

Tedaj je

$$u = Av \iff v = A^{-1}u = Bu$$

Če to razpišemo, smo izpeljali:

IZREK (Formula inverzije) Naj bosta $A = (a_{n,k})$ in $B = (b_{n,k})$ matriki povezovalnih koeficientov dveh polinomskih zaporedij in (u_n), (v_n) kompleksni zaporedji. Tedaj je

$$\boxed{u_n = \sum_{k=0}^{n} a_{n,k} v_k \ (\forall n) \iff v_n = \sum_{k=0}^{n} b_{n,k} u_k \ (\forall n).}$$

Binomska inverzija: Kot vemo sta (x^n) in $((x-1)^n)$ povezani takole,

$$x^n = \sum_{k=0}^{n} \binom{n}{k} (x-1)^k \quad \text{in} \quad (x-1)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^k .$$

Formula inverzije zato implicira, da je

$$\boxed{u_n = \sum_{k=0}^{n} \binom{n}{k} v_k \ (\forall n) \iff v_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} u_k} \quad \text{Binomska inverzija} \quad (1)$$

Če za zaporedje v vzamemo $(-1)^k v_k$, tedaj se binomska inverzija prepiše v

$$u_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k v_k \iff (-1)^n v_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} u_k$$

$$\boxed{(-1)^n \cdot (-1)^{n-k} = (-1)^{2n-k} = (-1)^{-k} = (-1)^k}$$

torej:

$$\boxed{u_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} v_k \iff v_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u_k}$$

Primer (uporabe binomske inverzije). Kot vemo (za preštevanje razdelitev glede na velikost slike) velja

$$r^n = \sum_{k=0}^{n} \binom{r}{k} k! S(n,k) \quad (r \geq 0)$$

$\boxed{r^n = \#\text{ vseh funkcij iz } n\text{-množice v } r\text{-množico}}$

$$u = (0^n, 1^n, 2^n, \dots) \qquad v = (0!\,S(n,0), 1!\,S(n,1), 2!\,S(n,2), \dots)$$

n fiksen. V (1) torej pišemo:

$$\underset{r^n}{\underset{\parallel}{u_r}} = \sum_{k=0}^{r} \binom{r}{k} \underset{k!S(n,k)}{\underset{\parallel}{v_k}} \iff v_r = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} u_k$$

$$\Rightarrow \quad r!\,S(n,r) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} k^n$$

$$\Rightarrow \quad \boxed{S(n,r) = \frac{1}{r!} \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} k^n}$$

Stirlingova inverzija

Kot že vemo, je

$$x^n = \sum_{k=0}^{n} S(n,k)\, x^{\underline{k}} \quad \text{in tudi} \quad x^{\underline{n}} = \sum_{k=0}^{n} (-1)^{n-k} s(n,k)\, x^k .$$

Zato formula inverzije pravi, da velja:

$$\boxed{u_n = \sum_{k=0}^{n} S(n,k)\, v_k \ (\forall n) \iff v_n = \sum_{k=0}^{n} (-1)^{n-k} s(n,k)\, u_k \ (\forall n)}$$

Stirlingova inverzija

Opomba: ker sta ustrezni matriki inverzni, odtod zastonj dobimo tudi:

$$\boxed{\sum_{\ell \geq 0} S(n,\ell)(-1)^{\ell-m} s(\ell,m) = \delta_{n,m}.}$$

n-ta vrstica × m-ti stolpec

Dosedanja strategija: rekurzije kombinatoričnih števil $\longrightarrow$ zveze med pol. zaporedji

Sedaj obratno: pol. zaporedja (z določenimi) komb. lastnostmi $\longrightarrow$ rekurzije med povezovalnimi koeficienti.

Def. Naj bo $\alpha = (a_1, a_2, \dots)$ zaporedje kompleksnih števil. Tedaj je

$$p_0^{(\alpha)}(x) = 1, \quad p_n^{(\alpha)}(x) = (x-a_1)(x-a_2)\cdots(x-a_n) \quad (n \geq 1)$$

ustrezno zaporedje polinomov za α.

Primeri:

$\alpha = (0,0,0,\dots) \longrightarrow (p_n^{(\alpha)}) = (x^n)$

$\alpha = (1,1,1,\dots) \longrightarrow (p_n^{(\alpha)}) = ((x-1)^n)$

$\alpha = (0,1,2,\dots) \longrightarrow (p_n^{(\alpha)}) = (x^{\underline{n}})$

$\alpha = (0,-1,-2,\dots) \longrightarrow (p_n^{(\alpha)}) = (x^{\overline{n}})$

$\alpha = (1,q,q^2,\dots) \longrightarrow (p_n^{(\alpha)}) = (g_n(x))$ ... Gaussovi polinomi

IZREK. Naj bo $\alpha=(a_1,a_2,\dots)$, $\beta=(b_1,b_2,\dots)$ in

$$p_n^{(\alpha)}(x)=\sum_{k=0}^{n} c_{n,k}\,p_k^{(\beta)}(x).$$

Tedaj velja:

$$c_{0,0}=1,\quad c_{0,k}=0\quad(k>0),$$

$$c_{n,0}=(b_1-a_1)(b_1-a_2)\dots(b_1-a_n)\qquad(n\geq 1),$$

$$c_{n,k}=c_{n-1,k-1}+(b_{k+1}-a_n)\,c_{n-1,k}\qquad(n,k\geq 1).$$

Dokaz. Poenostavimo zapis: $p_n(x)=p_n^{(\alpha)}(x)$, $q_n(x)=p_n^{(\beta)}(x)$. Tedaj $p_n(x)=\sum_{k=0}^{n} c_{n,k}\,q_k(x)$.

$$1=p_0(x)=c_{0,0}\underbrace{q_0(x)}_{=1}\Rightarrow c_{0,0}=1.$$

- če je $k>0$ je seveda $c_{0,k}=0$ (saj je $(c_{n,k})$ spodnje trikotna matrika).
- Izračunajmo $p_n(x)$ v b_1:

$$p_n(b_1)=\sum_{k=0}^{n} c_{n,k}\underbrace{q_k(b_1)}_{=0\ \forall k\geq 1}=c_{n,0}.\quad\text{Torej } p_n(b_1)=(b_1-a_1)\dots(b_1-a_n)=c_{n,0}$$

- $p_n(x)=(x-a_n)\,p_{n-1}(x)$ in $q_{k+1}(x)=(x-b_{k+1})\,q_k(x)$. Zato je

$$\boxed{x\,q_k(x)=q_{k+1}(x)+b_{k+1}q_k(x)}$$

$$\begin{aligned}
p_n(x)&=x\cdot p_{n-1}(x)-a_n p_{n-1}(x)\\
&=\sum_{k\geq 0} c_{n-1,k}\,x\,q_k(x)-a_n\sum_k c_{n-1,k}\,q_k(x)\\
&=\sum_k c_{n-1,k}\,q_{k+1}(x)+\sum_k c_{n-1,k}\,b_{k+1}\,q_k(x)-\sum_k a_n c_{n-1,k}\,q_k(x)\\
&=\sum_k c_{n-1,k-1}\,q_k(x)+\dots\\
&=\sum_k\big(c_{n-1,k-1}+(b_{k+1}-a_n)\,c_{n-1,k}\big)\,q_k(x)\\
&=\sum_k c_{n,k}\,q_k(x).
\end{aligned}$$

$\square$

<u>Primeri</u> (vse naše prejšnje rekurzije so pokrite s tem okvirom).

1. $\alpha = (0,0,0,\dots)$ $\quad \beta = (1,1,1,\dots)$ $\quad : \quad p_n^{(\alpha)}(x) = x^n \quad p_n^{(\beta)}(x) = (x-1)^n$

   $\Rightarrow \quad C_{n,0} = 1 \quad , \quad C_{n,k} = C_{n-1,k-1} + C_{n-1,k} \longrightarrow \binom{n}{k}$

2. $\alpha = (0,0,0,\dots)$ $\quad \beta = (0,1,2,\dots)$ $\quad (b_n = (n-1))$

   $C_{n,0} = 0 \quad , \quad C_{n,k} = C_{n-1,k-1} + k\, C_{n-1,k} \longrightarrow S(n,k)$

3. $\alpha = (0,-1,-2,\dots)$ $\quad \beta = (0,0,0,\dots)$ $\quad (a_n = -(n-1))$

   $C_{n,0} = 0, \quad C_{n,k} = C_{n-1,k-1} + (n-1)\, C_{n-1,k} \longrightarrow s(n,k)$

4. $\alpha = (0,0,0,\dots)$ $\quad \beta = (1,q,q^2,\dots)$ $\quad (b_n = q^{n-1})$

   $C_{n,0} = 1, \quad C_{n,k} = C_{n-1,k-1} + q^k\, C_{n-1,k}. \longrightarrow \left[{n \atop k} \right]_q$