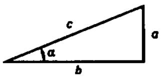
TRIGONOMETRIČNE FUNKCIJE



V pravokotnem trikotniku so:
k a t e t i – stranici a in b ob pravem kotu,
h i p o t e n u z a – stranica c nasproti pravemu kotu.

Trigonometrične funkcije kota α so razmerja stranic pravokotnega trikotnika:

sinus	$\sin\alpha = a/c$	tangens	$\tan\alpha = a/b$
kosinus	$\cos\alpha = b/c$	kotangens	$\cot\alpha = b/a$

Vrednosti trigonometričnih funkcij dobimo z računalnikom (ali iz ustreznih preglednic).

Vrednosti trigonometričnih funkcij pomembnejših kotov

$\alpha =$ ° / rad	0° 0	30° $\pi/6$	45° $\pi/4$	60° $\pi/3$	90° $\pi/2$	180° π	270° $3\pi/2$	360° 2π
$\sin\alpha =$	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	1	0	-1	0
$\cos\alpha =$	1	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	-1	0	1
$\tan\alpha =$	0	$\frac{1}{3}\sqrt{3}$	1	$\sqrt{3}$	$\pm\infty$	0	$\pm\infty$	0
$\cot\alpha =$	$\pm\infty$	$\sqrt{3}$	1	$\frac{1}{3}\sqrt{3}$	0	$\pm\infty$	0	$\pm\infty$

Trigonometrične funkcije v različnih področjih kotov

$\varphi =$ ° / rad	$\pm\alpha$ $\pm\alpha$	$90° \pm \alpha$ $(\pi/2) \pm \alpha$	$180° \pm \alpha$ $\pi \pm \alpha$	$270° \pm \alpha$ $(3\pi/2) \pm \alpha$	$360° \pm \alpha$ $2\pi \pm \alpha$
$\sin\varphi =$	$\pm\sin\alpha$	$+\cos\alpha$	$\mp\sin\alpha$	$-\cos\alpha$	$\pm\sin\alpha$
$\cos\varphi =$	$+\cos\alpha$	$\mp\sin\alpha$	$-\cos\alpha$	$\pm\sin\alpha$	$+\cos\alpha$
$\tan\varphi =$	$\pm\tan\alpha$	$\mp\cot\alpha$	$\pm\tan\alpha$	$\mp\cot\alpha$	$\pm\tan\alpha$
$\cot\varphi =$	$\pm\cot\alpha$	$\mp\tan\alpha$	$\pm\cot\alpha$	$\mp\tan\alpha$	$\pm\cot\alpha$

Osnovne odvisnosti trigonometričnih funkcij

$\sin^2\alpha + \cos^2\alpha = 1$ $\qquad \sin\alpha = \sqrt{1-\cos^2\alpha}$

$\tan\alpha = \sin\alpha/\cos\alpha$ $\qquad \cos\alpha = \sqrt{1-\sin^2\alpha}$

$\cot\alpha = \cos\alpha/\sin\alpha$ $\qquad 1+\tan^2\alpha = 1/\cos^2\alpha$

$\tan\alpha\cot\alpha = 1$ $\qquad 1+\cot^2\alpha = 1/\sin^2\alpha$

Trigonometrične funkcije dveh kotov

$$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\tan(\alpha\pm\beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$$
$$\cot(\alpha\pm\beta) = \frac{\cot\alpha\cot\beta\mp 1}{\cot\beta\pm\cot\alpha}$$

Za $\alpha = \beta$ velja:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$\tan 2\alpha = 2\tan\alpha/(1-\tan^2\alpha)$$
$$\cot 2\alpha = (\cot^2\alpha - 1)/2\cot\alpha$$

Nadalje velja:

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$
$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$
$$2\sin^2\alpha = 1-\cos 2\alpha$$
$$2\cos^2\alpha = 1+\cos 2\alpha$$
$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\tan\alpha \pm \tan\beta = \frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$$
$$\cot\alpha \pm \cot\beta = \frac{\sin(\beta\pm\alpha)}{\sin\alpha\sin\beta}$$
$$2\sin\alpha\sin\beta = -\cos(\alpha+\beta) + \cos(\alpha-\beta)$$
$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$
$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

1. Dokaz za inverznost:

F(G(y)) = y
G(F(x) = x

2. Limita zaporedja:

- zaporedja:

$$A(x,y) = \frac{x+y}{2} \geq G(x,y) = \sqrt{x\cdot y} \geq H(x,y) = \frac{}{\frac{1}{x}}$$

- primeri:

* $a_n = \frac{2}{n} \quad n \geq 1 \qquad 2,\ 1,\ \frac{2}{3}, \frac{1}{2}, ...$

1° Zaporedje je padajoče:

$$a_{n+1} - a_n = \frac{2}{n+1} - \frac{2}{n} = \frac{2n-2n-2}{(n+1)n} = -\frac{2}{(n+1)n} <$$

2° Omejeno navzdol: $a_n = \frac{2}{n} > 0$

3° Konvergira: $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{2}{n} = 0$

* $a_0 = 0 \qquad a_{n+1} = \frac{1}{3}a_n^2 + \frac{2}{3}$

$$1°\ \text{Dokaz, da je } 0 \leq a \leq 1: \ \text{matematičaa indukcija}$$
$$\text{(i) n} = 1 \Rightarrow a_1 = \frac{1}{3}\cdot 0 + \frac{2}{3} = \frac{2}{3} < 1$$
$$\text{(ii) velja za } a_n < 1, \text{dokaz za } a_{n+1} < 1: a_{n+1} = \frac{1}{3}$$
$$2°\ \text{Zaporedje je naraščaraše}: a_{n+1} - a_n = \frac{1}{3}a_n^2 + $$
$$3°\ \text{Limita zap}: \lim_{n\to\infty} a_n = a$$
$$\lim_{n\to\infty}(a_{n+1}) = \lim_{n\to\infty}(\frac{1}{3}a_n^2 + \frac{2}{3}) \Rightarrow a = \frac{1}{3}a^2 + $$

* Indukcija po korakih:

$$0 < a_1 < 1 \qquad a_{n+1} = 1 - \sqrt{1-a_n}$$
$$1° Dokaz: 0 < a_n < 1$$
$$\text{(i) n} = 1 \Rightarrow 0 < a_1 < 1$$
$$\text{(ii) } 0 < a_n < 1 \Rightarrow 0 > -a_n > -1 \Rightarrow 1 > 1 - a_n > 0 \Rightarrow 1 > \sqrt{1-a}$$
$$2°\ \text{Padajoce zap}: a_{n+1} - a_n = 1 - \sqrt{1-a_n} - a_n = \frac{((1-a_n) - \sqrt{1}}{(1-}$$
$$3°\ \lim_{n\to\infty} a_n = a \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}(1-\sqrt{1-a_n}) \Rightarrow a = 1 - \sqrt{1-}$$
$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1-\sqrt{1-a_n}}{a_n} = \lim_{n\to\infty}\frac{1-1+a_n}{a_n(1+\sqrt{1-a_n})} = \text{li}$$

3. Vrste:

* $$\sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + ...$$

$$S_n = \sum_{k=1}^{\infty}\frac{1}{(2k-1)(2k+1)} \Rightarrow \frac{1}{(2k-1)(2k+1)} = \frac{a}{2k-1} + \frac{b}{2k+1} =$$
$$S_n = \sum_{k=1}^{\infty}(\frac{1}{2(2k-1)} - \frac{1}{2(2k+1)}) = \frac{1}{2}(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - ... +$$
$$S = \lim_{n\to\infty} S_n = \frac{1}{2}\lim_{n\to\infty}(1 - \overbrace{\frac{1}{2n+1}}^{\to 0}) = \frac{1}{2}$$

- <u>Primerjalni kriterij:</u>

$$\sum_{k\to 1}^{\infty} a_k = \text{konvergira, če } a_k \leq b_k \ \ in \ \sum_{k\to 1}^{\infty} b_k \ \ konvergira. \ Isto \ velja$$

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$$\sum_{k=1}^{\infty}\frac{1}{k^2} \Rightarrow \frac{1}{k^2} \leq \frac{1}{k(k-1)} \ \ in \ \sum_{k=2}^{\infty}\frac{1}{k(k-1)} \ \ konvergira. \Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^2}$$

* $$\sum_{k=1}^{\infty}\frac{1}{1+a^k} \qquad a > 0$$

$$\text{Če je } a > 1 \Rightarrow \frac{1}{1+a^k} \leq \frac{1}{a^k} \quad in \quad \sum_{k=1}^{\infty} \frac{1}{a^k} \quad je\ konv$$

$$\text{Če je } a = 1 \Rightarrow \sum_{k=1}^{\infty} \frac{1}{1+1^k} = \sum_{k=1}^{\infty} \frac{1}{2} - divergira$$

$$\text{Če je } 0 < a < 1 \Rightarrow \lim_{k \to \infty} \frac{1}{1+a^k} = 1 - \text{ni izpolnjen p}$$

- Kvocientni kriterij:

$$\sum_{k=1}^{\infty} a_k , \qquad a_k > 0 \qquad l = \lim_{k \to \infty} \frac{a_{k+1}}{a_k} \Rightarrow \text{če je} \ldots \text{če} \ldots$$

*

$$\sum_{k=1}^{\infty} \frac{k}{(k+1)!} \Rightarrow l = \lim_{k \to \infty} \frac{\frac{k+1}{(k+2)!}}{\frac{k}{(k+1)!}} = \lim_{k \to \infty} \frac{(k+1)(k+}{(k+1)!(k+}$$

- Korenski kriterij:

$$\sum_{k=1}^{\infty} a_k \qquad a_k > 0 \qquad l = \lim_{k \to \infty} \sqrt[k]{k^k} \Rightarrow \text{če je 0} \ldots \text{če je} \ldots$$

*

$$\sum_{k=1}^{\infty} \frac{1}{(\log k)} \Rightarrow l = \lim_{k \to \infty} \sqrt[k]{\frac{1}{(\log k)^k}} = \lim_{k \to \infty} \frac{1}{\log k} = 0$$

Korenski in kvocientni kriterij za $l = 1$ nista definirana.

- Cauchyjev kriterij:

$$\sum_{k=1}^{\infty} a_k \text{ konvergira takrat, ko } \sum_{k=1}^{\infty} 2^k a_{2^k} \text{ konvergi}$$

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$$\sum_{k=1}^{\infty} \frac{1}{k} \qquad a_k = \frac{1}{k} \Rightarrow a_{2^k} = \frac{1}{2^k} \qquad \Rightarrow \sum_{k=0}^{\infty} 2^k a_{2^k} = $$

- Alternirajoče vrste:

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k \text{ - konvergira, ko } a_k \text{ padajoc in } \lim_{k \to}$$

*

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k(k+2)} \qquad a_k = \frac{1}{k(k+2)} \geq a_{k+}$$

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} \frac{1}{k(k+2)} = 0 - konvergira$$

$$S_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k(k+2)} \Rightarrow \frac{1}{k(k+2)} = \frac{a}{k} + \frac{b}{k+2}$$

$$S_n = \frac{1}{2} \sum_{k=1}^{n} (-1)^{k+1} \left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{1}{2} \left((1 - \frac{1}{3}) - (\frac{1}{2} - \right.$$

$$S = \lim_{n \to \infty} S_n = \frac{1}{2} \left(1 - \frac{1}{2}\right) = \frac{1}{4}$$

Vrsta $\sum_{k=1}^{\infty} (-1)^{k+1} a_{2^k}$ je absolutno konvergentna, če je vrsta $\sum_{k=1}^{\infty} a_k$ konvergentna.