VEČKRATNI INTEGRALI

Volumen: $V = \iiint\limits_{a} dx \cdot dy \cdot dz$

Uporabljaj enačbe dane v pogojih!

Nove spremenljivke:

$$\Phi_{(u,v,w)} = (\Phi_1, \Phi_2, \Phi_3)$$

Jacobss:

$$J_\Phi = \det \begin{vmatrix} \Phi_{1u} & \Phi_{1v} & \Phi_{1w} \\ \Phi_{2u} & \Phi_{2v} & \Phi_{2w} \\ \Phi_{3u} & \Phi_{3v} & \Phi_{3w} \end{vmatrix} =$$

$$= (\Phi_{1u} \cdot \Phi_{2v} \cdot \Phi_{3w}) + (\Phi_{1v} \cdot \Phi_{2w} \cdot \Phi_{3u}) + (\Phi_{1w} \cdot \Phi_{2u} \cdot$$

$$- (\Phi_{1w} \cdot \Phi_{2v} \cdot \Phi_{3u}) - (\Phi_{1v} \cdot \Phi_{2u} \cdot \Phi_{3w}) - (\Phi_{1u} \cdot \Phi_{2w} \cdot$$

Integral:

$$\iiint\limits_{\Phi(Q)} f(x, y, z) dx \cdot dy \cdot dz = \iiint\limits_{Q} f(x(u, v, w), y$$

$$\cdot \left| J_\Phi(u, v \right.$$

STANDARDNE FUNKCIJE ZA UVEDBO NOVE SPREMENLJIVKE:

Vstavljaj r-e in φ-je v začetne enačbe pa boš dobu meje!

-<u>Polarne koordinate</u>:

Φ(r,φ) = (r · cosφ, r · sinφ)

x = r · cos φ

y = r · sin φ

$J_\Phi = r$

Če središče ni v centru in je v (a,b), izberi:

(x – a) = r · cos φ, (y – b) = r · sin φ, $J_\Phi = r$

-<u>Cilindrične koordinate</u>:

x = r · cos φ

y = r · sin φ

z = z

$J_\Phi = r$

-<u>Krogelne koordinate</u>:

x = r · sin θ · cos φ

y = r · sin θ · sin φ

z = r · cos θ

$J = r^2 \cdot \sin\theta$

$\varphi \subseteq (0, 2\pi)$, $\theta \subseteq (0, \pi)$

Standardni integrali:

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$$\int_a^b \sin^2 \varphi \cdot d\varphi = \frac{1}{2} \int_a^b (1 - \sin 2\varphi) d\varphi = \frac{1}{2} (\varphi - \frac{\sin 2}{2}$$

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$$\int_a^b \cos^3 \varphi \cdot d\varphi = \int_a^b (1 - \sin^2 \varphi) \cos \varphi \cdot d\varphi = (u - \frac{u^3}{3}$$

$$nova\ \ spre: \quad \sin \varphi = u \Longrightarrow \cos \varphi \cdot d\varphi = du$$

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$$\int_a^b \sin^4 \theta \cdot d\theta = \frac{1}{4} \int_a^b (\frac{3}{2} - 2\cos 2\theta + \frac{1}{2} \cos 4\theta) d\theta =$$

$$= \frac{1}{4} (\frac{3}{2} \theta - 2 \sin 2\theta \cdot \frac{1}{2} + \frac{1}{2} \sin 4\theta \cdot \frac{1}{4}) \Big|_a^b$$

f(x)	F(x)
x^n	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	log x + C
a^x	$\frac{a^x}{\log x} + C$
e^{ax}	$\frac{e^{ax}}{a} + C$
sin x	- cos x + C
cos x	sin x + C
$\frac{1}{\sin^2 x}$	- ctg x +C
$\frac{1}{\cos^2 x}$	tg x + C
tg x	- log(cos x) + C
ctg x	log(sin x) + C
$\frac{1}{\sqrt{a^2 - x^2}}$	arcsin $\frac{x}{a}$ + C
$\frac{1}{\sqrt{x^2 + a^2}}$	$\operatorname{arcsin} h \frac{x}{a} + C = \log(x +$
$\frac{1}{\sqrt{x^2 - a^2}}$	$\operatorname{arccos} h \frac{x}{a} + C = \log(x +$
$\frac{1}{x^2 + 1}$	arctg x + C
sinh x	cosh x + C
cosh x	sinh x + C
$\frac{1}{\sin x}$	$\ln(tg \frac{x}{2}) + C$
$\frac{1}{\sinh x}$	$\ln(tgh \frac{x}{2}) + C$
$\frac{1}{(x-a)^n}$	$-\frac{1}{(n-1)(x-a)^{n-1}}$
$\frac{1}{x^2 - 1}$	$\frac{1}{2} \log(\frac{x-1}{x+1}) + C$

<u>Integriranje po delih</u>:

$$\int F(x) \cdot g(x) dx = F(x) \cdot G(x) - \int f(x) \cdot G(x) dx$$

Primeri:

$$(I) \qquad \int \log x dx = \int 1 \cdot \log x dx = x \cdot \log x - \int x \cdot \frac{1}{x}$$

$$(II) \qquad \int_0^1 (1-x^2)e^{-2}dx = (1-x^2)(-\frac{1}{2}e^{-2x})\Big|_0^1 -$$

$$\frac{1}{2} - (x(-\frac{1}{2}e^{-2x})\Big|_0^1 - \int_0^1 -\frac{1}{2}e^{-2x}dx) = \frac{1}{2} + \frac{1}{2}e^{-2}$$

<u>Nova spremenljivka</u>: Primeri

$$(I) \qquad \int (1\text{-}\mathrm{x}^2)^2 x dx = (u = x^2, du = 2xdx) = \int ($$

$$(II) \qquad \int_{-1}^{1} \frac{dx}{\sqrt{x^2+x+1}} = \int_{1}^{-1} \frac{\mathrm{dx}}{\sqrt{(x+1/2)^2+3/4}} = ($$

$$\int_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{du}{(1+u)^2} = \operatorname{arcsin} hu\Big|_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} = ...$$

Ko integriramo racionalne funkcije, dobimo čelne : $\int \frac{Ax+B}{(x^2+ax+b)^m}dx$, kjer

kvadratni polinom v imenovalcu ne moremo razstaviti. Vedno zapišemo:

$$x^2 + ax + b = (x+\frac{a}{2})^2 + (b - \frac{a^2}{4}) =$$

$= (x+\frac{a}{2})^2 + c^2$, kjer damo novo spremenljivko: $x + \frac{a}{2} = c \cdot u$

<u>Integracija racionalnih funkcij</u>: <u>razcep na parcialne ulomke</u>: Primeri:

$$(I) \qquad f(x) = \frac{1}{x^2(x^2+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{(1+x^2)}$$

$$1 = Ax(1+x^2)^2 + B(1+x^2)^2 + (Cx+D)x^2(1+x^2)$$

$$\mathrm{x}^5 : A+C=0; \ \mathrm{x}^4 : B+D=0; \ \mathrm{x}^3 : 2A+C+E=$$

$$\text{Razberemo} : A=0, B=1, C=0, D=-1, E=0, F=$$

$$\text{Sledi} : \frac{1}{x^2(x^2+1)^2} = \frac{1}{x^2} - \frac{1}{(1+x^2)} - \frac{1}{(1+x^2)^2} =$$

<u>Integrali racionalni izrazov</u>:

Vedno deluje nova spremenljivka: $u = \tan\frac{x}{2} \Rightarrow dx = \frac{2du}{1+u^2}$, in iz tega sledi , da je:

$\sin x = \frac{2u}{1+u^2} \quad \text{in} \quad \cos x = \frac{1-u^2}{1+u^2}$ *Obstaja tudi*:

$$u = \tanh\frac{x}{2}, du = \frac{1-u^2}{2}dx, \sinh x = \frac{2u}{1-u^2}$$

<u>Izlimitirani integrali</u>: Primeri:

$$(I) \qquad \int_{-\infty}^{\infty} e^{-x^2} e^{-(a-x)^2} dx = e^{-a^2} \int_{-\infty}^{\infty} e^{-2x^2+2ax} = e^{-}$$

$$(\sqrt{2}(x - \frac{a}{2}) = u, \sqrt{2}dx = du) = e^{-\frac{a^2}{2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{\sqrt{2}}$$