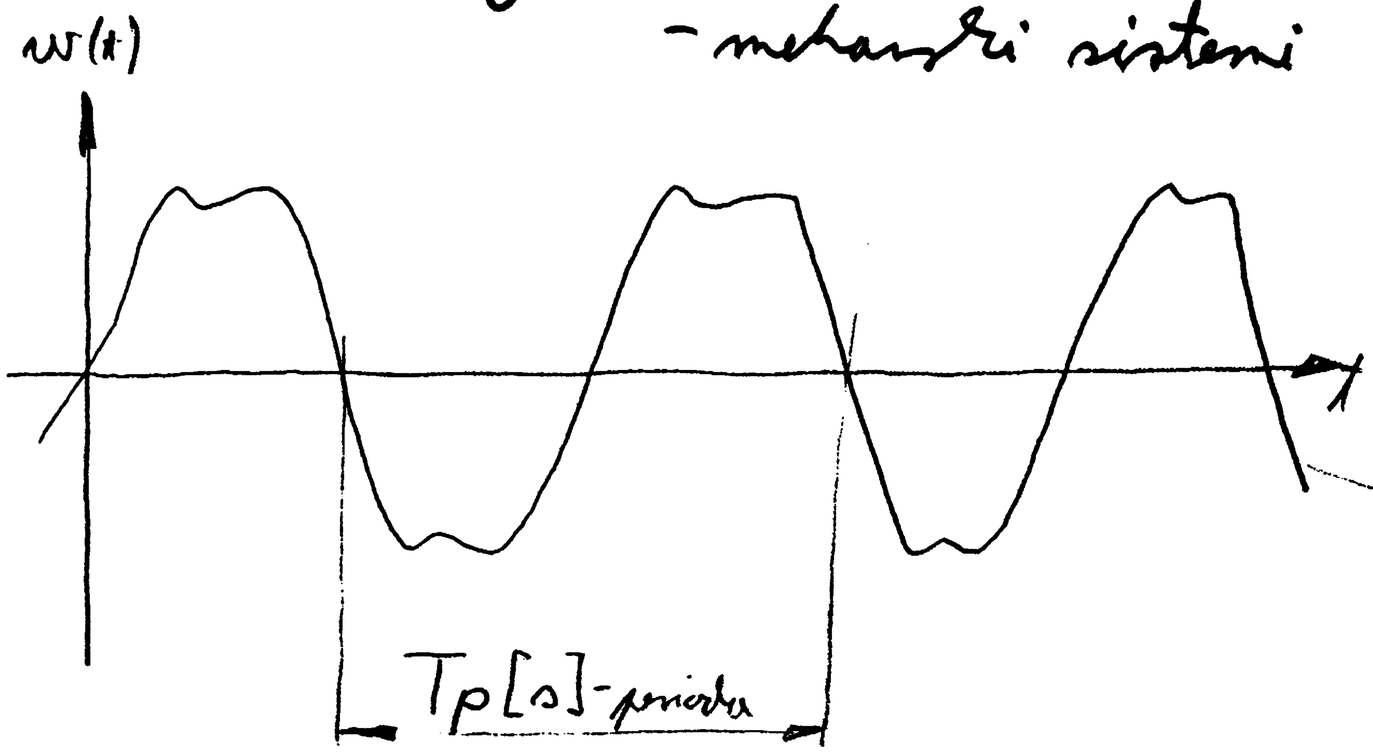
ODZIV SISTEMA Z N=1 NA PERIODIČNO MOTNJO

Tvori motnje - batni motorji
- mehanski sistemi



w(t) je periodična motnja

FOURIER je dokazal da lahko x(t) popišemo z neko vrsto

$$x(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos \omega t + \dots + b_1 \sin \omega t + b_2 \sin \omega t$$

srednja vrednost funkcije — osnovni ton

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cdot \cos \omega t + b_n \sin \omega t)$$

a_n, b_n so koeficienti fourierove vrste

$$a_0 = \frac{2}{T_p} \int_0^{T_p} x(t)\, dt$$

$$a_n = \frac{2}{T_p} \int_0^{T_p} x(t) \cos n\omega t\, dt$$

$$b_n = \frac{2}{T_p} \int_0^{T_p} x(t) \sin n\omega t\, dt$$

$\omega = \frac{2\pi}{T_p}$ osnovna fundamentalna frekvenca

$$x(t) = c_0 \sum_{n=1}^{\infty} C_n \cdot \cos(n\omega t - \Phi_n)$$

$C_n = \sqrt{a_n^2 + b_n^2}$ - amplituda

$\operatorname{tg} \Phi_n = \frac{b_n}{a_n}$ - fara

$F(t)$ je periodična funkcija

Iščemo $x(t)$ v ustaljenem stanju

$$m\ddot{x} + d\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \omega \sin n\omega t + b_n \omega \sin n\omega t$$

$$x = x_p(t)$$

Zaradi linearnosti

1.) $m\ddot{x} + d\dot{x} + kx = \frac{a_0}{2}$

2.) $m\ddot{x} + d\dot{x} + kx = a_n \overset{\cos}{\cancel{\sin}} n\omega t$ $\quad > n = 1, \dots, \infty$

3.) $m\ddot{x} + d\dot{x} + kx = b_n \cos n\omega t$

$$x_1 = \frac{a_0}{2k}$$

$$x_2(t) = \frac{a_0}{k} \frac{1 \cdot \cos(n\omega t - \varphi_n)}{\sqrt{\left(1 - \left(\frac{\omega n}{\omega_0}\right)^2\right)^2 + \left(2\delta\left(\frac{n\omega}{\omega_0}\right)\right)^2}}$$

$$x(t) = x_p(t) = x_1(t) + x_2(t) + x_3(t)$$

$$m\ddot{x} + d\dot{x} + kx = F_0 \cdot \sin \omega t$$

$$x(t) = X \sin(\omega t - \varphi)$$

$$X = \frac{F_0}{k} \beta$$

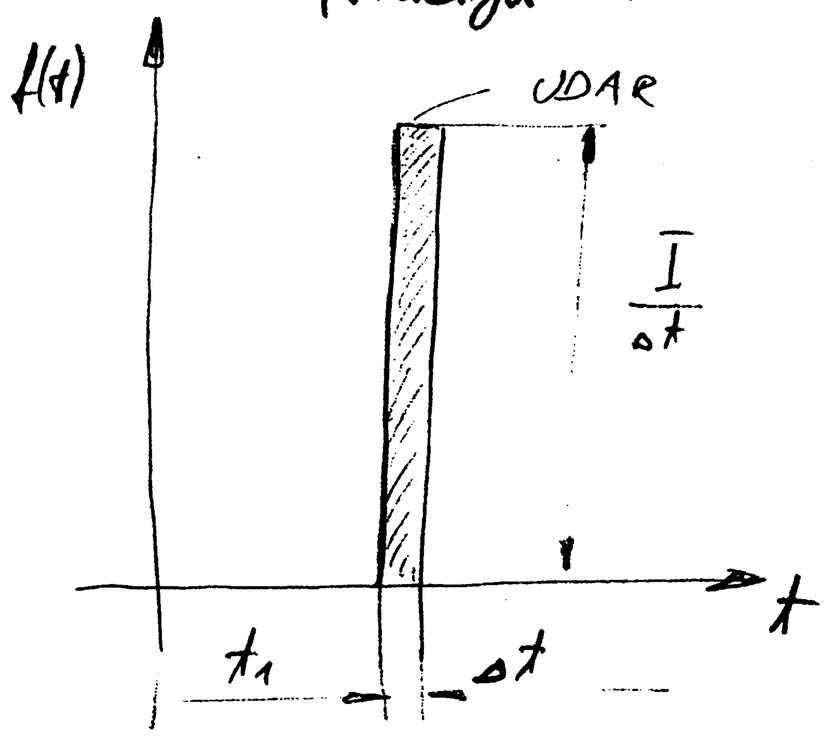
$$\operatorname{tg} \varphi = \frac{2\delta \frac{\omega}{\omega_0}}{1 - \left(\frac{\omega}{\omega_0}\right)^2}$$

$$\beta = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(2\delta \frac{\omega}{\omega_0}\right)^2}}$$

NIHANJE LINEARNEGA SISTEMA ZARADI UDARNE MOTNJE

UPORABA KONVOLUCIJSKEGA INTEGRALA

Definicija enotskega impulza ; Dirac-ova delta funkcija



$\Delta t \to 0 \qquad \frac{I}{\Delta t}$ → proti vsem mejam

V resnici je Δt neznano.

$$\delta(t-t_1) = \begin{cases} 0 & \text{za } t \neq t_1 \\ \infty & \text{za } t = t_1 \end{cases}$$

Integral produkta Diracove delta funkcije s poljubno $f(t)$

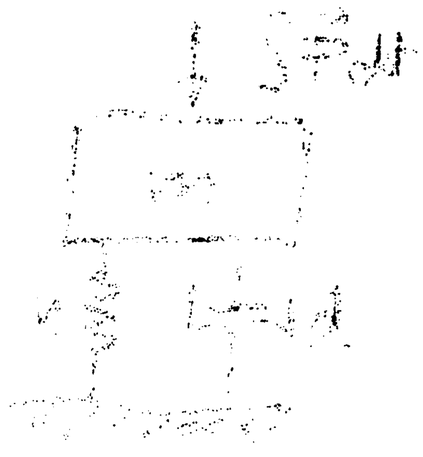
$$\int_0^\infty f(t)\,\delta(t-t_1)\,dt = f(t) \qquad 0 < t_1 < \infty$$

$$I = \Delta\vec{p} = m\vec{v}_2 - m\cancel{\vec{v}_1}^{\,0}$$ – sistem na začetku miruje

gibalna količina

$$\vec{I} = \int_0^t \vec{F}(t)\,dt \qquad \vec{v}_2 = \frac{\vec{I}}{m}$$

PRIMER VZBUJANJA OSCILATORJA Z IMPULZOM



Začetni pogoji

$$x(0) = 0$$
$$\dot{x}(0) = \frac{I}{m}$$

NEDUŠEN OSCI. $d=0$

$$m\ddot{x} + kx = 0$$

$$x(t) = A \cdot \cos(\omega_0 t) + B \sin(\omega_0 t) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$A = 0 \qquad B = \frac{I}{m\omega_0}$$

odziv je nadalje

$$x(t) = \frac{I}{m\omega_0} \sin(\omega_0 t)$$

Impulzna prenosna funkcija

$$g(t) = \frac{x(t)}{I} = \frac{1}{m\omega_0} \sin(\omega_0 t)$$

DUŠEN OSCILATOR

δ = razmerje dušenja

$$\delta = \frac{d}{d_{kri}} \qquad d_{kri} = 2m\omega_0$$

$$m\ddot{x} + d\dot{x} + kx = 0$$

$$x(t) = e^{-\delta\omega_0 t}\left[A \cdot \cos(\omega_{od} t) + B \cdot \sin(\omega_{od} t)\right]$$

$$A = 0 \qquad B = \frac{I}{m\omega_{od}}$$

odziv je nadalje:

$$x(t) = \frac{I}{m\omega_{od}} e^{-\delta\omega_0 t} \sin(\omega_{od} t)$$

Impulzna prenosna funkcija

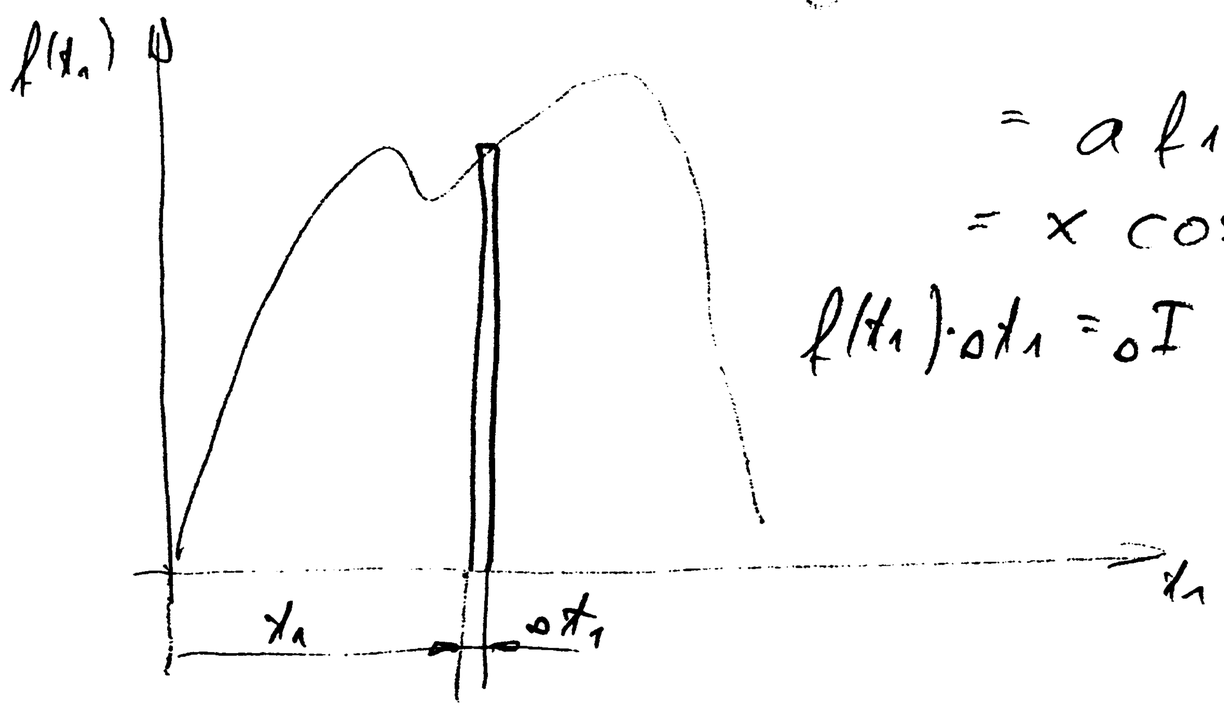
$$g(t) = \frac{1}{m\omega_{od}} e^{-\delta\omega_0 t} \sin(\omega_{od} t)$$

KONVOLUCIJSKI INTEGRAL

Potrebne lastnosti sistema za uporabo konvolucijskega integrala so:

- linearnost
- časovna invariantnost
- kavzalnost

$$\Delta x(t) = f(t_1)\,\Delta t_1\, g(t-t_1)$$



$$= a f_1(t) + b f_2(t)$$

$$= x\cos a \cdot x(t_1) + b \cdot x(t_2)$$

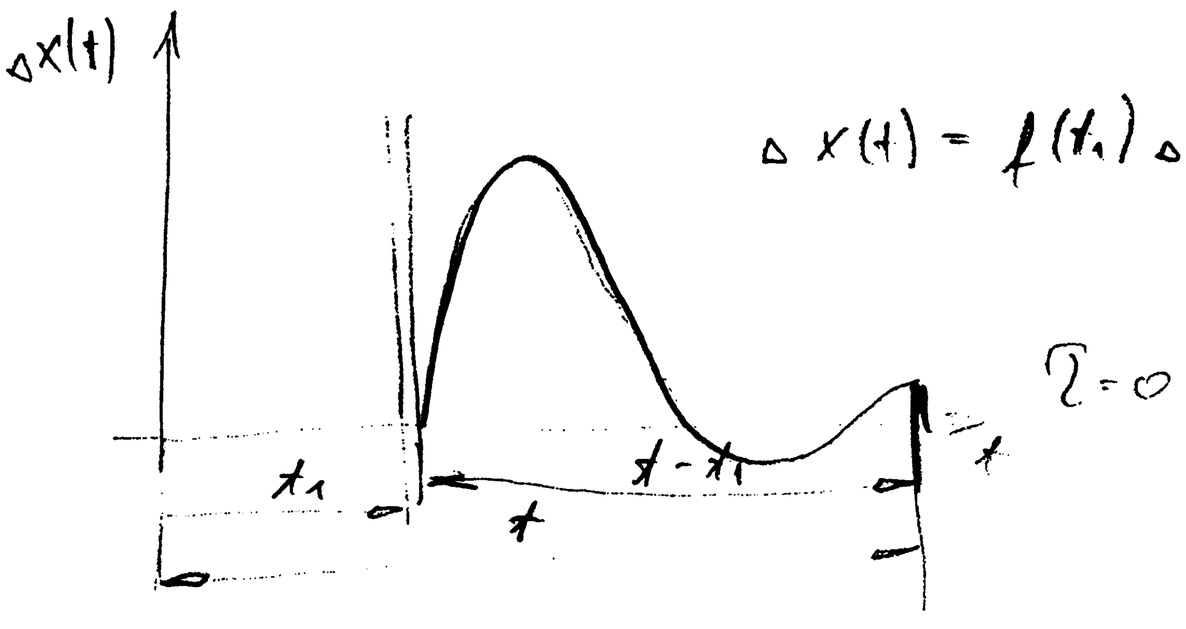
$$f(t_1)\cdot\Delta t_1 = \Delta I$$

$$g(t) = \frac{\Delta x(t)}{I}$$

$$\tau = t - t_1$$

Z uporabo superpozicije dobimo

$$x(t) = \lim_{\Delta i \to 0} \sum_{i=1}^{n} f(t_i)\, g(t-t_i)\,\Delta t_i = \underbrace{\int_0^t f(t_1)\, g(t-t_1)\, dt_1}_{\text{KONVOLUCIJSKI INTEGRAL}}$$



$$\Delta x(t) = f(t_1)\,\Delta t_1\, g(t-t_1)$$

Zapis konvolucije s pomočjo spominske spremenljivke τ

Splošno lahko zapišemo

$$x(t) = \int_{-\infty}^{+} f(t_1)\, g(t-t_1)\, dt_1$$

razširitev

KAZALNI SISTEM sistem se lahko začne odzivati šele po motnji ali kakšni drugi dani motnji

Sedaj uvedemo novo časovno spremenljivko

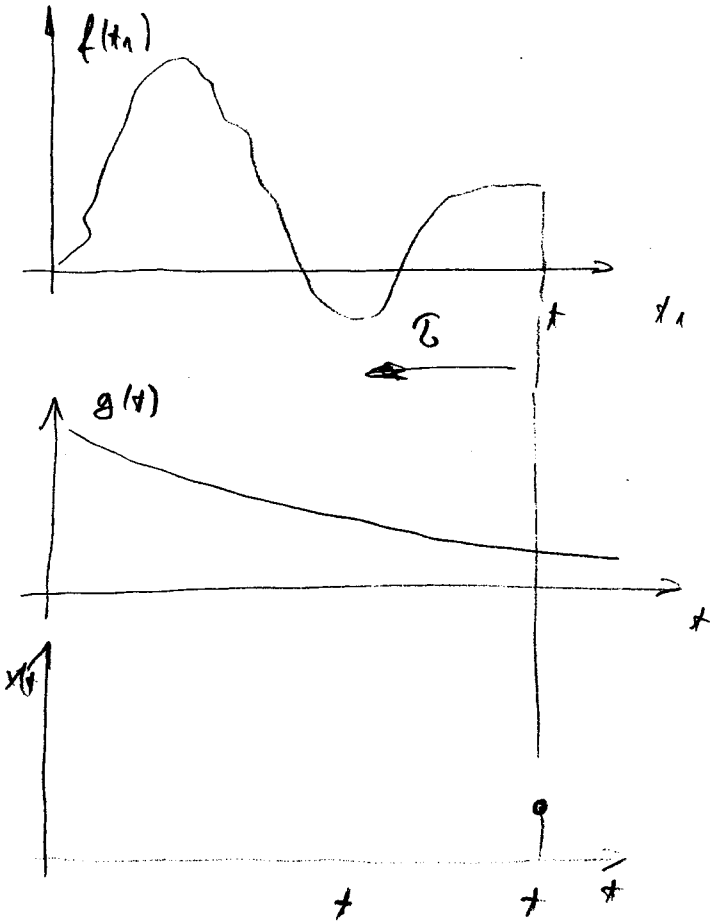
$$\tau = t - t_1 \qquad d\tau = -dt_1$$

$$x(t) = \int_{\infty}^{0} f(t-\tau)\, g(\tau)\, d\tau$$

$$x(t) = \int_{0}^{t} f(t-\tau)\, g(\tau)\, d\tau$$

τ - spominska časovna spremenljivka

$g(\tau)$ - dinamični spomin sistema (oscilatorja)



$$x(t) = \int_{0}^{t} f(t_1)\, g(t-t_1)\, dt_1$$

drugi način

$$x(t) = \int_{0}^{t} f(t-\tau)\, g(\tau)\, d\tau$$

Konvolucija je vsota vsotnih produktov

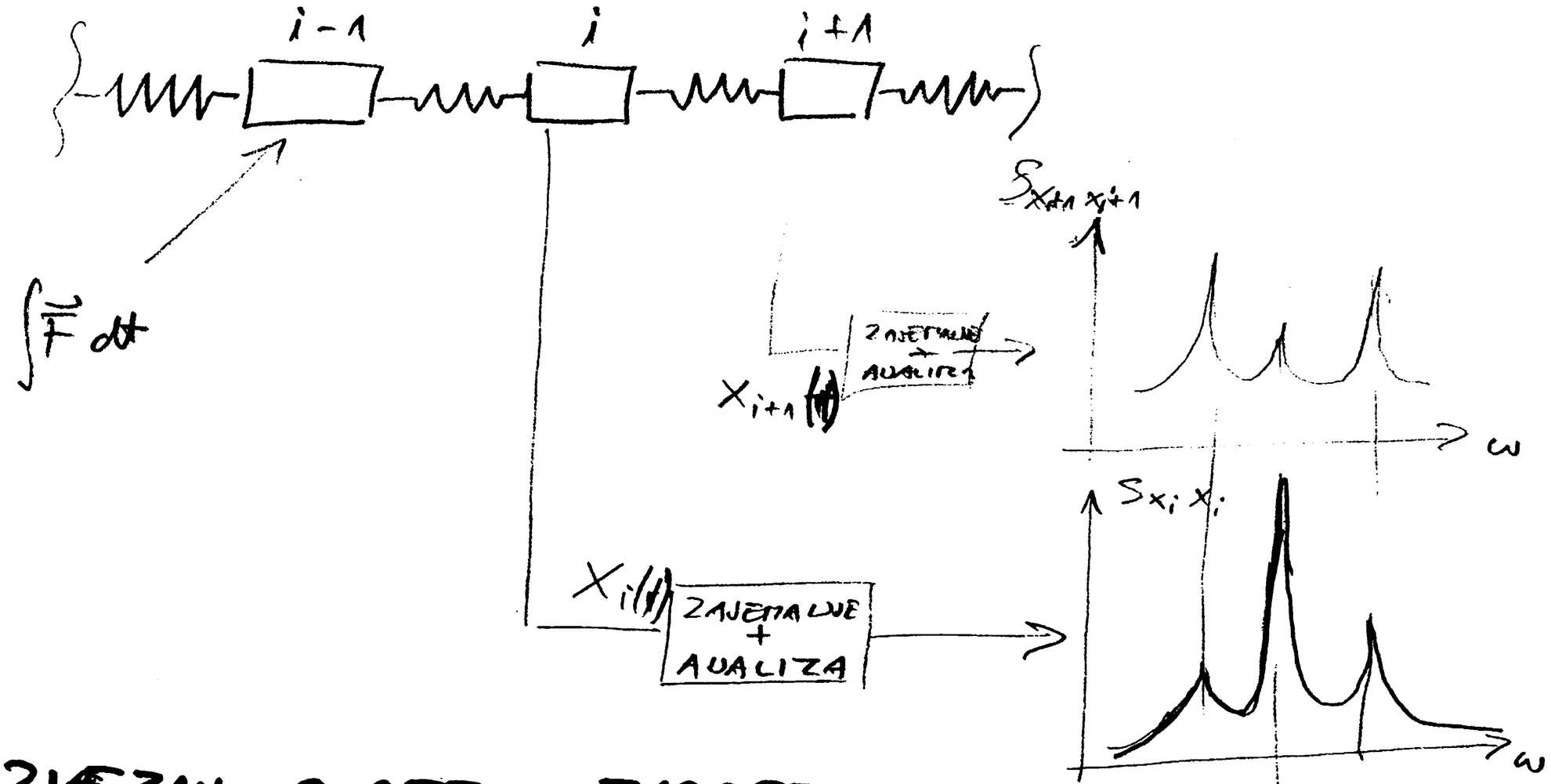
$$x(t) = \int_{0}^{t} f(t-\tau)\, g(\tau)\, d\tau = \lim_{\Delta\tau_i \to 0} \sum_{i=0}^{n} f(t - i\Delta\tau)\, g(i\Delta\tau)\, \Delta\tau$$

$$= f(t)\, g(0)\, \Delta\tau + f(t-\Delta\tau)\, g(\Delta\tau)\, \Delta\tau + f(t-2\Delta\tau)\, g(2\Delta\tau)\, \Delta\tau + f(t-3\Delta\tau)\, g(3\Delta\tau)\, \Delta\tau$$

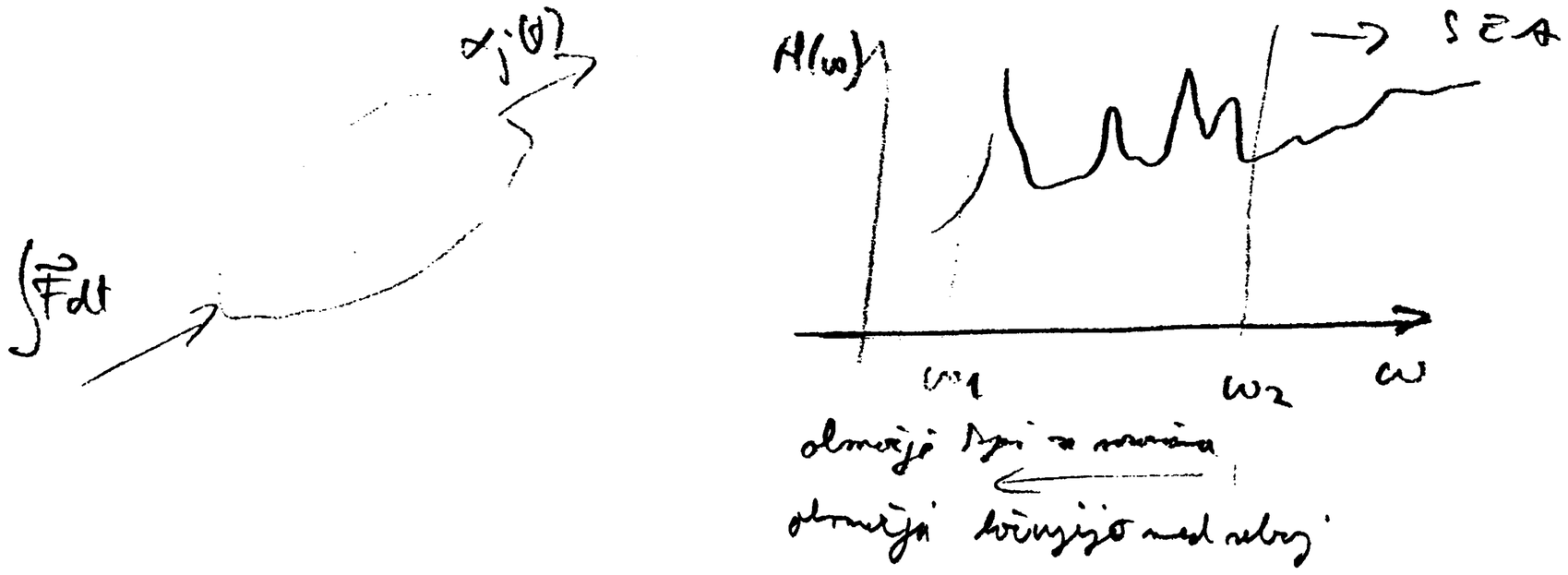
$$x(t) = \left[f(t)\, g(0) + f(t-\Delta\tau)\, g(\Delta\tau) + \cdots \right] \Delta\tau$$

UPORABA UDARNE MOTNJE

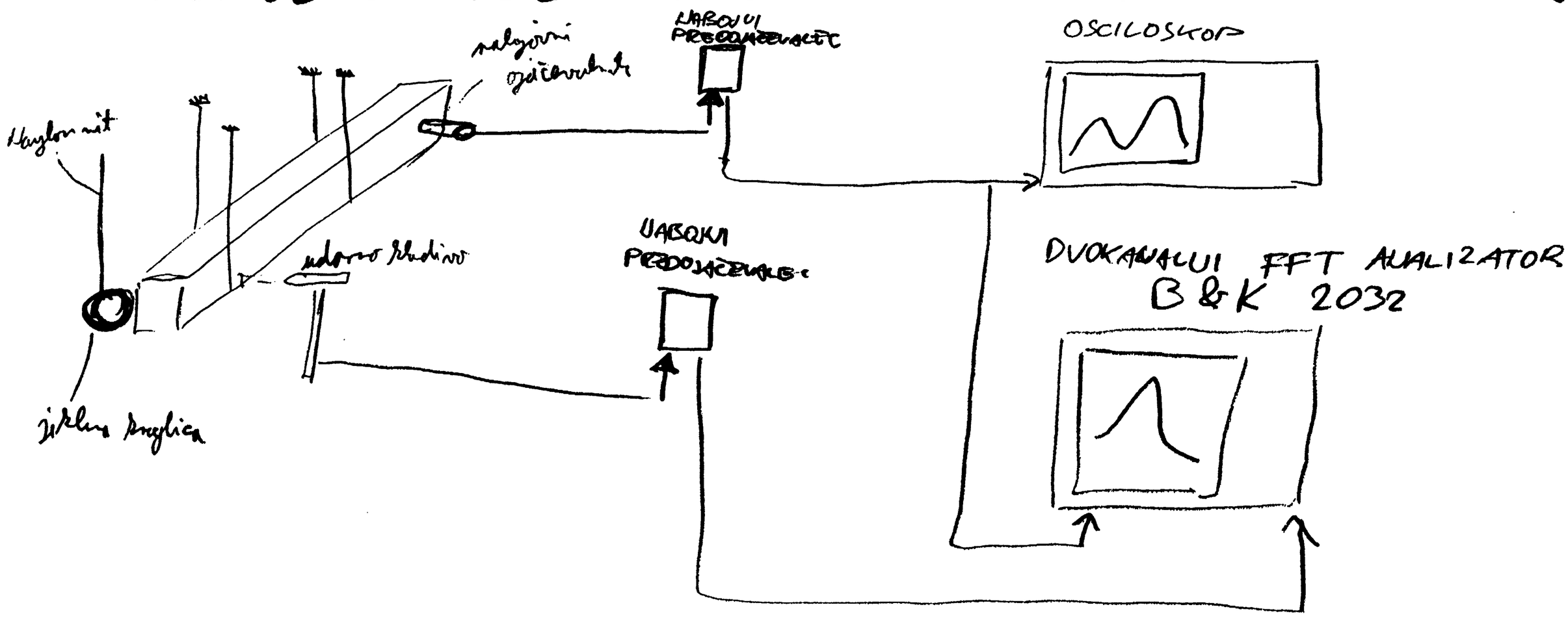
DISKRETNI SISTEMI



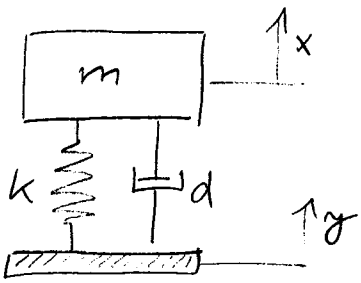
ZVEZNI SISTEM – EKSPERIMENTALNA STRUKTURNA DIN.



NEPORUŠNO TESTIRANJE POŠKODOVANIH STRUKTUR



ODZIV NA VZBUJANJE PODLAGE



Gibalna enačba

$$m\ddot{x} = -k(x-y) - d(\dot{x}-\dot{y})$$

Vpeljemo novo spremenljivko $z = x - y$

$$m\ddot{z} + d\dot{z} + k\dot{z} = -m\ddot{y}$$

Upoštevamo podobnost med enačbami

$$m\ddot{z} + d\dot{z} + kz = F$$

Za podkritično vistvano dušeno sistem je impulzna prenosna funk.

$$g(t) = \frac{1}{m\omega_{0d}} e^{-\delta\omega_0 t} \sin(\omega_{0d} t)$$

Rešitev gibalne enačbe

$$x(t) = \int_0^t f(t_1)\, g(t-t_1)\, dt_1$$

$$z(t) = -\frac{1}{\omega_{0d}} \int_0^t \ddot{y}(t_1)\, e^{-\delta\omega_0 (t-t_1)} \sin(\omega_{0d}(t-t_1))\, dt_1$$

NIHANJE SISTEMOV Z VEČ PROSTOSTNIMI STOPNJAMI

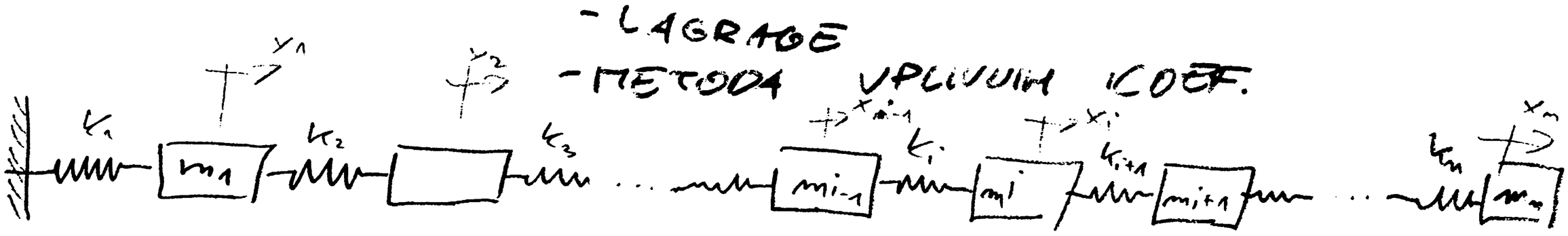
$N > 1$ - št. koordinat, ki so neodvisne od drugih - št prost. stopenj.

NAČELOMA : Imamo toliko ω_{0i} $\{\bar{X}\}^{(i)}$; $i = 1 \dots N$ kolikor je koordinat.

ZELO : Obvladovanje velikih sistemov „velik prost. stopenj - zvezni sistemi."

1) POSTAVIMO GIBALNE ENAČBE

- NEWTON
- LAGRANGE
- METODA VPLIVNIH KOEF.



a) [illegible] ned $x_{i+1} > x_i$ $\quad i = 1, N-1$

i-ta masa

$$m_i \ddot{x}_i = -k_i (x_i - x_{i-1}) + k_{i+1} (x_{i+1} - x_i) + F_i(t) \quad i = 1 \dots N$$

SISTEM UREDIMO

Vsi členi ki imajo koordinate $(x_i \dots)$ in njihove poljube damo na levo

$$m_i \ddot{x}_i - k_i x_{i-1} + (k_i + k_{i+1}) x_i - k_{i+1} x_{i+1} = F(t) \quad i = 1 \dots N$$

Izračunamo najprej MASNO MATRIKO

$$[M] = \begin{bmatrix} m_{11} & m_{12} & \dots & m_{1N} \\ m_{21} & m_{22} & & \vdots \\ \vdots & \vdots & \ddots & m_{N-1,N} \\ m_{N1} & m_{N2} & \dots & m_{NN} \end{bmatrix}$$ - Masna matrika

Če je masna matrika diagonalna ⇒ je mavni sistem ~~[illegible]~~ neveznan. (v vsaki diferencialni enačbi rešujemo samo eno mesto)

VEKTOR POSPEŠKOV

$$\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_i \\ \vdots \\ \ddot{x}_N \end{Bmatrix}$$ – VEKTOR POSPEŠKOV

TOGOSTNA MATRIKA SISTEMA

$$[k] = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1N} \\ k_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ k_{N1} & \cdots & \cdots & k_{NN} \end{bmatrix}$$

Princip recipročnosti [matrika je simetrična] $k_{ij} = k_{ji}$

VEKTOR POMIKOV

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$$

VEKTOR ZUNANJIH SIL

$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_N \end{bmatrix}$$

za lastna nihanja $\{F\} = \{0\}$

ENAČBA NIHANJA

$$[M]\{\ddot{x}\} + [k]\cdot\{x\} = \{F\}$$

METODA VPLIVNIH KOEFICIENTOV

PROŽNOSTNE vplivne koeficiente
TOGOSTNE vplivne koeficiente

PROŽNOSTNI VPLIVNI KOEFICIENTI

V sistemu sila F_j na mestu j, x_{ij} - pomik na mestu i
x_{ij} - pomik na mestu i zaradi ~~sile~~ sile na mestu j

$$x_{ij} = \underbrace{p_{ij}}_{\text{prožnostni vplivni koeficient}} F_j$$

$$x_i = \sum_{j=1}^{n} p_{ij} F_j$$

$$\{x\} = [p] \cdot \{F\}$$

TOGOSTNI VPLIVNI KOEFICIENT

F_i - sila na mestu i, x_j

$$F_{ij} = \underbrace{k_{ij}}_{\text{togostni vplivni koeficient}} x_j$$

$$F_i = \sum_{j=1}^{n} k_{ij} x_j \qquad i = 1 \dots n \qquad \Rightarrow [p] = [k]^{-1}$$

$$\{F\} = [K] \cdot \{x\} = [K] \cdot [P] \cdot \{F\}$$

DOLOČITEV LASTNIH VREDNOSTI

$$[M]\cdot\{\ddot{x}\}+[K]\cdot\{x\}=\{0\}$$

Iščemo $x_i = x_i(t)$; $i=1\dots N$, ki ob začetnih pogojih $x_i(0)$; $\dot{x}_i(0)$; $i=1\dots N$ rešijo zgornjo ~~enačbo~~ enačbo.

OSNOVNA IDEJA

$$x_i(t) = \underbrace{X_i}_{} \cdot T(t)$$

$$x_j(t) = X_j \cdot T(t)$$

amplituda pomika na i-tem mestu

$$\left\langle \frac{x_i(t)}{x_j(t)} \right\rangle \neq f(t)$$ – nihanja ni sinhrona

IDEJA: ločitev časa od prostora

$$\{X\} = \begin{Bmatrix} X_1 \\ X_2 \\ X_N \end{Bmatrix} \quad \{x\} = \{X\}\,T(t)$$

$$[M]\{X\}\,\ddot{T}+[K]\{X\}\,T=0$$

v ~~skalarnem~~ tenzorskem zapisu

$$\sum_{j=1}^{N} m_{ij} X_j \ddot{T} = -\sum k_{ij} X_j \; T = 0$$

$$\left(\sum_{j=1}^{N} m_{ij} X_j\right)\ddot{T} = -\left(\sum_{j=1}^{N} k_{ij} X_j\right)\cdot T$$

$$\frac{\ddot{T}}{T} = -\frac{\sum_{j=1}^{N} k_{ij} X_j}{\sum_{j=1}^{N} m_{ij} X_j} = \text{konst} = -\omega^2$$

$\ddot{T} + \omega^2 T = 0$ ← homogena enačba drugega reda

$T(t) = A \cdot \cos \omega t + B \sin \omega t$; AB sta konstanti; f (začetni pogoji)

$$\sum_{j=1}^{n} (k_{ij} - \omega^2 m_{ij}) \bar{X}_j = 0$$

Matrično

$$([K] - \omega^2 [M]) \{\bar{X}\} = 0$$

$$(k_{11} - \omega^2 m_{11}) \bar{X}_1 + k_{12} \bar{X}_2 + \dots + k_{1n} \bar{X}_n = 0$$

za diagonalno mora matrika

$$k_{n1} \bar{X}_1 + \dots\dots + (k_{nn} - \omega^2 m_{nn}) \bar{X}_n = 0$$

$$[M] - \omega^2 [M] \{\bar{X}\} = 0 \quad / \cdot [M]^{-1}$$

$$\det([A] - \lambda [I]) = 0$$

enotna matrika

$$[A] = [M^{-1}] \cdot K$$

dinamična matrika sistema

$$a^2 = \lambda$$

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & & \vdots \\ & & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} - \lambda \end{vmatrix}$$

Lastne vrednosti so realne vrednosti

$P(\lambda^n) = 0$ ← polinom stega reda → n realnih korenov

prva lastna vrednost — $\lambda_1 < \lambda_2 < \dots < \lambda_n$ — lastne vrednosti sistema

$\omega_{0i} = \sqrt{\lambda_i}$ $\quad i = 1, \dots, n$ lastne frekvence sistema

področje delovanja turbo strojev

DOLOČITEV LASTNIH VEKTORJEV

$$([A]-\lambda[I])\{X^{(i)}\}=0$$ $\{X^{i}\}$ - vektor amplitud lastna oblika

$$z_{11}X_1^{(i)}+z_{12}X_2^{(i)}+\dots+z_{1N}X_N=0$$ $\{X^{(i)}\}$ - i-ti lastni vektor

$\vdots$

$$z_{N1}X_1^{(i)}+\dots\qquad z_{NN}X_N=0$$

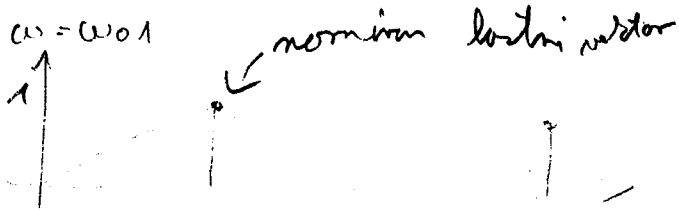
izračunamo N razmerij

$$\{X^{(i)}\}=\begin{Bmatrix}X_1^{(i)}\\X_2^{(i)}\\\vdots\\X_N^{(i)}\end{Bmatrix}=X_i^{(i)}\qquad \begin{Bmatrix}1\\X_2/X_1\\X_3/X_1\\\vdots\\X_N/X_1\end{Bmatrix}=X_1^{(i)}\{X^{(i)}\}$$

normiran lastni vektor

Lastni vektor predstavlja razmerja pomikov!

PRIKAZ LASTNIH VEKTORJEV



Prvi lastni vektor nam pove razmerja amplitud, ko se nosilec giblje s prvo frekvenco

ORTOGONALNOST LASTNIH VEKTORJEV

$$[M]\cdot\{\ddot{x}\}+[k]\{x\}=\{0\}$$

$$-\omega^2\cdot[M]\cdot\{X\}+[k]\cdot\{X\}=0$$

$$[K]\cdot\{X^{(i)}\}=\lambda[M]\cdot\{X^{(i)}\} \quad /\{X^{(j)}\}$$

$$\{X^{(j)}\}^T[k]\{X^{(i)}\}=\lambda_i\{X^{(j)}\}^T\cdot[M]\{X^{(i)}\} \quad (A)$$

$$\{X^{(i)}\}^T[k]\{X^{(j)}\}=\lambda_j\{X^{(i)}\}^T[M]\{X^{(j)}\} \quad (B)$$

KER STA $[M]$ + $[K]$ SIMETRIČNA

$$\{X^{(j)}\}^T[k]\{X^{(i)}\}=\{X^{(i)}\}^T[k]\{X^{(j)}\}$$

A-B

$$0=(\lambda_i-\lambda_j)\{X^{(i)}\}^T[M]\{X_j\}$$

$$\boxed{\{X^{(i)}\}^T[M]\{X^{(j)}\}=0} \quad i\neq j$$

$i=1\ldots n$

$j=1\ldots n$

ORTOGONALNOST LASTNIH VEKTORJEV

$$\{X^{(i)}\}^T[k]\{X^{(j)}\}=0$$

PREHOD V GLAVNE KOORDINATE

$$\underbrace{\{x(t)\}}_{\times 1\times 1} = \underbrace{[\phi]}_{1\times 0} \cdot \{\eta(t)\}$$

$\{\eta(t)\}$ → glavne koordinate, modalne koordi.

$[\phi]$ → modalne koordinate

$\{x(t)\}$ → fizikalne koordinate

$$[\phi]_{1\times 0} = \left[\{X^{(1)}\}, \{X^{(2)}\}, \dots, \{X^{(0)}\}\right]$$

$$[M]\cdot\{\ddot{x}\} + [k]\{x\} = 0$$

$$[\ddot{x}] = [\Phi]\cdot\{\ddot{\eta}\}$$

$$[M][\Phi]\{\ddot{\eta}\} + [k][\Phi]\{\eta\} = \{0\} \qquad / [\Phi]^T$$

$$[\Phi]^T[M][\Phi]\{\ddot{\eta}\} + [\Phi]^T[k][\Phi]\{\eta\} = 0$$

$$\Rightarrow [\Phi]^T[k][\Phi] = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_0 \end{bmatrix}$$ – generirana togostna matrika sis.

$$[\Phi]^T[M][\Phi] = \begin{bmatrix} M_1 & 0 & 0 \\ 0 & M_2 & 0 \\ 0 & 0 & M_0 \end{bmatrix}$$ – generirana masna matrika sistema

DEFINIRAMO:

$$k_i = \{X^{(i)}\}^T[k]\{X^{(i)}\}$$

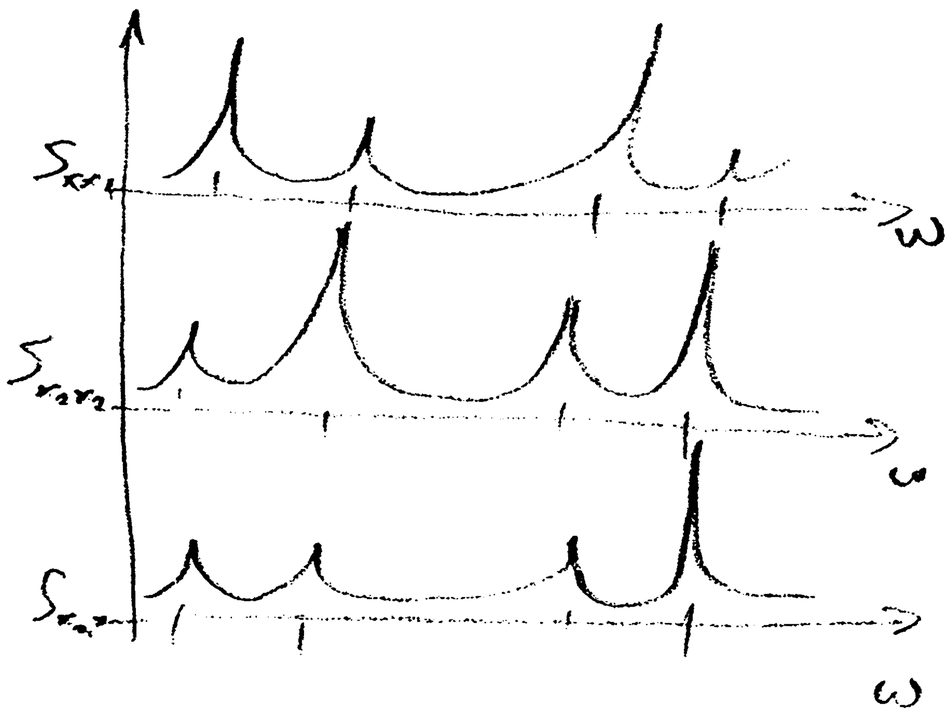
$$M_i = \{X^{(i)}\}^T[M]\{X^{i}\}$$

$$[M_0]\{\ddot{\eta}\} + [k_0]\{\eta\} = \{0\}$$

$\omega_{0i} = \sqrt{\frac{k_i}{M_i}}$ i-ta modalna koordinata vzbuja i-to lastno frekvenco

$$\{x(0)\} \qquad \{\dot{x}(0)\}$$

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_i(t) \\ \vdots \\ x_n(t) \end{Bmatrix} = \begin{bmatrix} & & & \\ X_1^{(1)} & X_i^{(2)} & \dots & X_i^{(n)} \\ & & & \end{bmatrix} \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \\ \vdots \\ \eta_i(t) \\ \vdots \\ \eta_n(t) \end{Bmatrix}$$



Vrhovi so vedno na istih mestih. Mešam ko se veličast glede na pozicijo mazinga in nöjo velikosti spreminja.

Vrh predstavljajo odziv sistema pri determiranih frekvencah, ki jih tem primeru lastna frekvenca.

VSILJENA NIHANJA SISTEMOV Z $N>1$ PROSTOSTIH STOPEN
(HARMONIČNO VZBUJANJE)

a) FIZIKALNE KOORDINATE
b) GLAVNE KOORDINATE

FIZIKALNE KOORDINATE

$$[M]\{\ddot{x}\}+[K]\{x\}=\{F\}=\{F_0\}\sin\omega t$$

↳ vektor vzbujalnih sil

$$\{F\}=\{F(t)\}=\begin{Bmatrix}F_{01}\\F_{02}\\\vdots\\F_{0N}\end{Bmatrix}\sin\omega t=\{F_0\}\sin\omega t$$

↳ vsiljena krožna frekvenca

Matematično je to sistem veznih nehomogenih diferencialnih enačb.

Iščemo partikularno rešitve

Nastavek za enačbo:

$$\{x(t)\}=\{X\}\sin\omega t$$

↳ vektor amplitud, vektor odzivov je nam

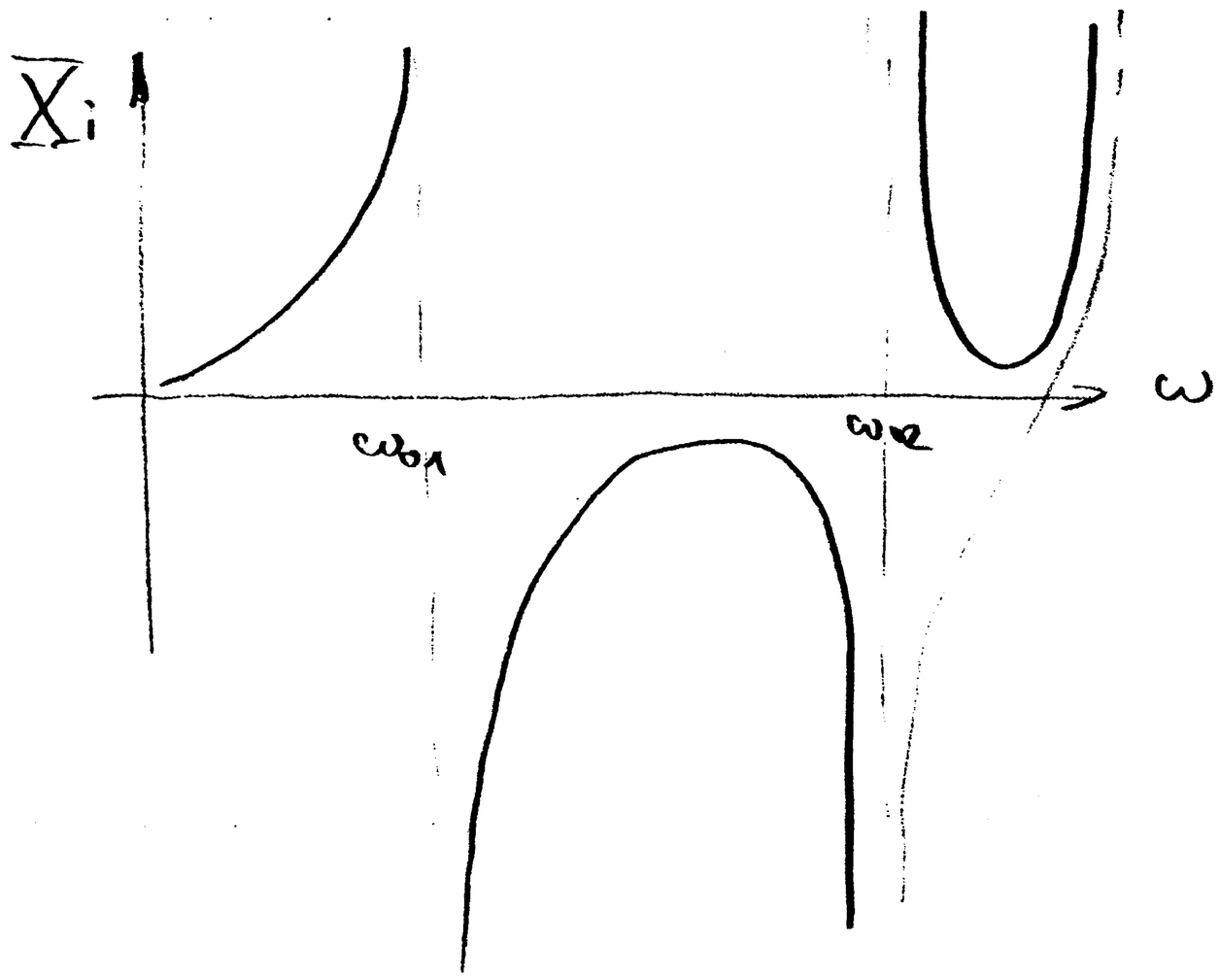
$$\{\ddot{x}\}=-\omega^2\{x\}$$

$$(-\omega^2[M]+[K])\{X\}\sin\omega t=\{F_0\}\sin\omega t$$

$$(-\omega^2[M]+[K])\{X\}=\{F_0\}$$

$$X_i=\frac{|D_i|}{|D|}\qquad i=1,\dots,N$$

Ko gre determinanta D proti 0, gredo odzivi $\to\infty$.
Rečemo, da resonanci odziv sistema gre proti neskončnosti.



Če pa imamo še dušenje „v praksi je vedno."

DUŠENJE - viskozni model
- suho drsno trenje
- strukturno dušenje

$$[M]\cdot\{\ddot{x}\} + \{DIS.\ EQ.\} + [K]\{x\} = \{F\}$$

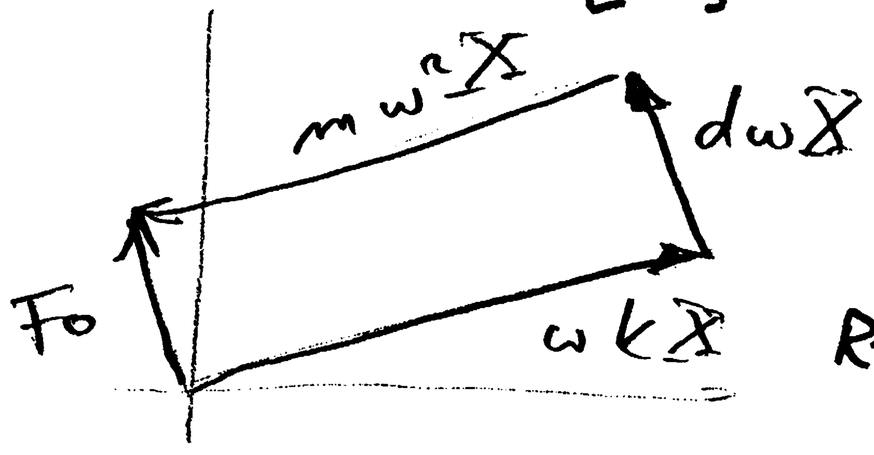
Za viskozno dušenje

$$+ [C]\cdot\{\dot{x}\}$$

$[D] \rightarrow$ dušilna matrika

Poenostavljen primer

$$[C] = \alpha[M] + \beta[K]$$ - RAYLEJKH-OV DUŠENJE



RESONANČNO STANJE

Ta graf nam pove da ima dušenje največji vpliv ravno v resonančnem področju. Torej pomeni da se izven resonačnega področja ne uporabljamo z dušenjem

$$\omega_{00} = \omega_0\sqrt{1-\delta^2}$$

inženirsko gledano ta enačbi

GLAVNE KOORDINATE

$$\{x\} = [\Phi]\cdot\{\eta\}$$

$$[m]\cdot\{\ddot{x}\} + [k]\{x\} = \{F\}$$

$$[m]\cdot[\Phi]\{\ddot{\eta}\} + [k][\Phi]\{\eta\} = \{F\} \quad \Big| \cdot [\Phi]^T$$

$$[m_g]\{\ddot{\eta}\} + [k_g]\{\eta\} = [\Phi]^T\{F\}$$

$$m_i\ddot{\eta} + k_i\eta = [X^{(i)}]^T\{F\} \quad \Big| : m_i$$

$$\ddot{\eta} + \frac{k_i}{m_i}\eta = Q_i \Rightarrow \text{ita posplošena zunanja sila}$$

$$\ddot{\eta} + \omega_{0i}^2\eta = Q_i$$

Partikularna rešitev

$$\eta_i(t) = E_i \sin\omega_{0i}t$$

$$E_i = \frac{\{X_i\}^T\{F\}}{X_i}\cdot\beta_i$$

$$\beta_i = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_{0i}}\right)^2\right)}}$$

Naš sistem je N razsežen in ima N prostostnih stopenj.

NIHANJE ZVEZNIH SISTEMOV

Osnovne lastnosti

masa, togost, dušenje so enako porazdeljeni.

∞ prostostnih stopenj
∞ lastnih oblik

Odniki so tovrstni kvazi funkcije tako kraja kot tudi časa

1.D SISTEMI (struna) - prečna nihanja, (palica) - osna nihanja, (gredi) - torzijska nihanja (nosilci) - prečna nihanja.

- prečna nihanja
- osna nihanja
- torzijska nihanja
- prečna nihanja nosilcu

upogib

$$\frac{\partial^2(\text{premik})}{\partial x^2} \propto \frac{\partial^2(\text{pospeška})}{\partial t^2}$$

$$w(x,t) \quad , \varphi(x,t)$$

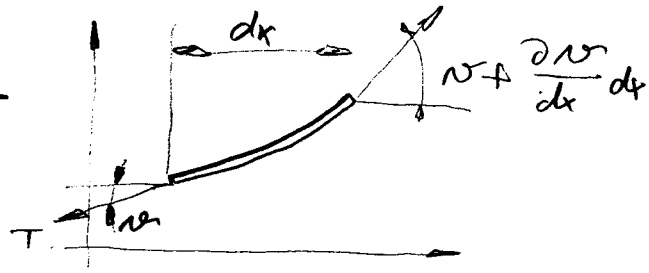
$$\frac{\partial^4 w}{\partial x^4} \propto \frac{\partial^2 w}{\partial t^2}$$

2.D SISTEMI membrana, plošča (pomembno za akustiko)
3.D SISTEMI oceani možgani.

LASTNA NEDUŠENA LINEARNA NIHANJA

PREČNA NIHANJA STRUNE
TORZIJSKA NIHANJA GREDI
RAVNINSKA UPOGIBNA NIHANJA

PREČNA NIHANJA STRUNE



PREDPOSTAVKE:

- $\eta = \frac{m}{l}$ masa na enoto dolžine
- kotiv so majhni
- osna sila je konstantna

NEWTONOV ZAKON

$$\sum F_{iy} = m\ddot{y}$$

$$-\cancel{Tv} + T\left(\cancel{v} + \frac{\partial v}{\partial x} dx\right) = \eta\, dx \cdot \frac{\partial^2 y}{\partial t^2}$$

$$v = \frac{\partial y}{\partial x}$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2}}$$

$$c^2 = \frac{T}{\eta}$$

→ hitrost širjenja valov

GIBALNA ENAČBA PREČNEGA NIHANJA STRUNE – VALOVNA ENAČBA

Iščemo $y = y(x,t)$ pri danih robnih pogojih

$y(0,t) = 0$

$y(l,t) = 0$

IŠČEMO $y = y(x,t)$

FOURIER-ov nastavek

$y = y(x,t) = X(x) \cdot T(t)$

$\frac{\partial^2 y}{\partial x^2} = X'' \cdot \ddot{T}$ → dvakrat odvedemo po času

$X''T = \frac{1}{c^2} X \ddot{T} \quad | : (XT)$

$\frac{X''}{X} = \frac{1}{c^2} \frac{\ddot{T}}{T} = \text{konst} = -\left(\frac{\omega}{c}\right)^2 \quad | \; c^2 \cdot T \quad | \; X$ → drugo področje

↳ prvo področje

L = D

$$\ddot{T} + \omega^2 T = 0$$

$T(t) = A \cdot \cos\omega t + B \sin\omega t$ → harmonično nihanje A, B sta funkcije začetnih pogojev

$$X'' + \left(\frac{\omega}{c}\right)^2 X = 0$$

$X(x) = C \cdot \cos\left(\frac{\omega}{c} x\right) + D \sin\left(\frac{\omega}{c} \cdot x\right)$ – splošna rešitev

C, D sta konstanti, f (robnih pogojev)

1.) $X(0) = 0 \Rightarrow 0 = C \cdot 1 + D \cdot 0 \Rightarrow C = 0$

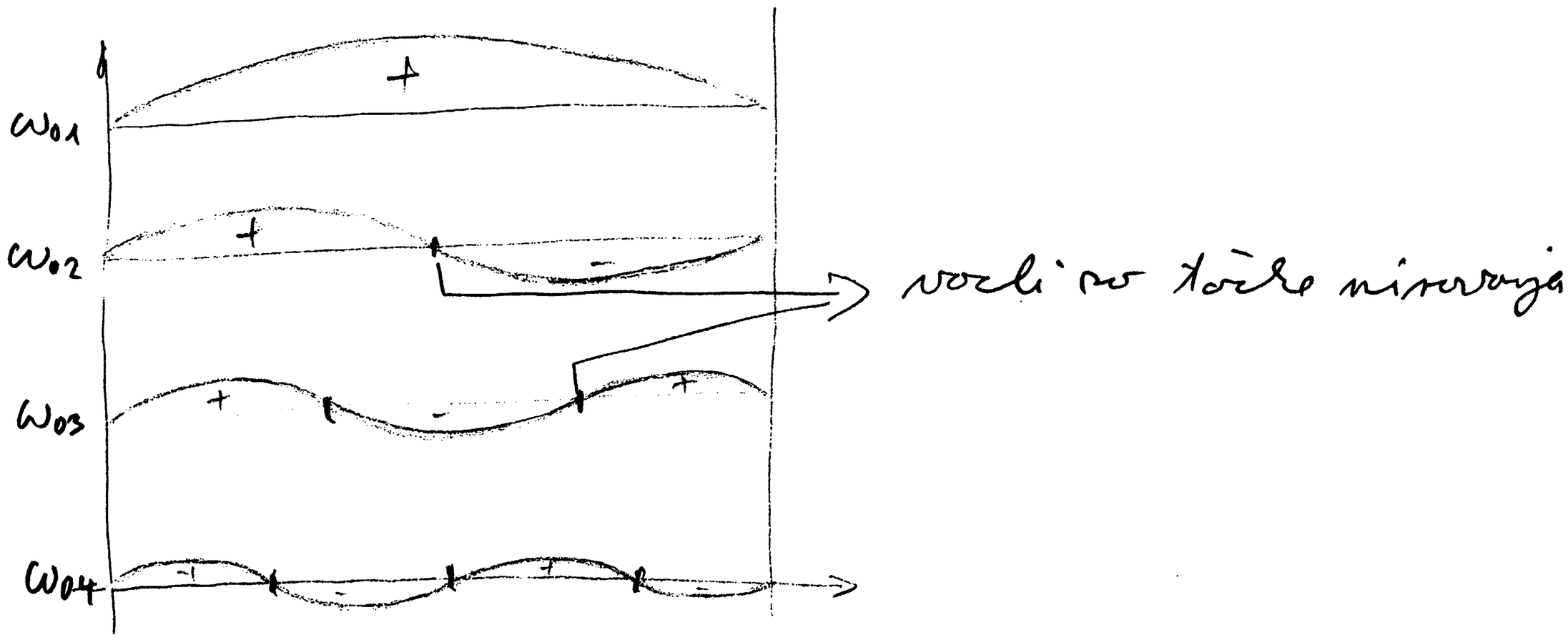
$$0 = D \cdot \underbrace{\sin\left(\frac{\omega}{c} \cdot l\right)}$$

$$\sin\left(\frac{\omega}{c} l\right) = n\pi \qquad n = 1, \dots, \infty$$

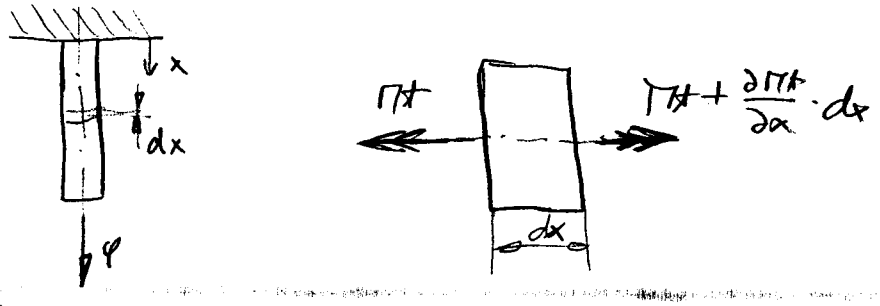
$\omega_{01} = \frac{\pi}{l}\sqrt{\frac{T}{\eta}}$ – prva lastna frekvenca BISTVENA

$\omega_{0n} = n\frac{\pi}{l}\sqrt{\frac{T}{\eta}}$ – n-ta lastna frekvenca

<u>**oblika**</u> $X_{(x)} = D_n \sin\left(\frac{\omega_{0n}}{c} \cdot x\right)$



TORZIJSKA NIHANJA GREDI



J - osni vztrajnostni moment kg m²

$$\sum M_t = J\ddot{\varphi}$$

$$-M_t + M_t + \frac{\partial M_t}{\partial x}dx = \frac{J}{l}dx \cdot \frac{\partial^2\varphi}{\partial t^2}$$

trdnost $-M_t = k_t \cdot \varphi = \frac{GI_t}{l}\varphi = GI_t \frac{\partial\varphi}{\partial x}$

$$\frac{\partial}{\partial x}\left(GI_t \cdot \frac{\partial\varphi}{\partial x}\right) = \frac{J}{l}\frac{\partial^2\varphi}{\partial t^2}$$

GI_t = konst

$$c^2 = \frac{GI_t \cdot l}{J}$$

$$c^2\frac{\partial^2\varphi}{\partial x^2} = \frac{\partial^2\varphi}{\partial t^2}$$

$$\varphi = \varphi(x,t) = \Phi(x)\cdot T(t)$$

$\vdots$

$$\Phi(x) = C\cdot\cos\left(\frac{\omega}{c}x\right) + D\sin\left(\frac{\omega}{c}x\right)$$

ROBNI POGOJI

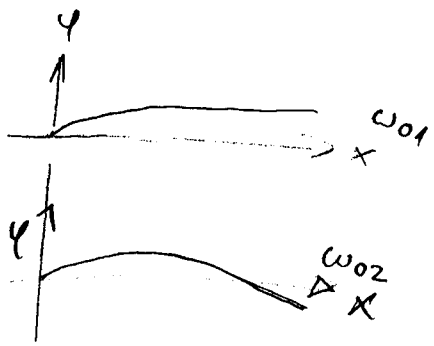
$\Phi(0) = 0$ $\quad x=0 \quad \Rightarrow C = 0$

$x = l \quad M_t(0) = 0 \quad \Phi'(x=l) = 0$

$$\Phi'(x) = D\frac{\omega}{c}\cos\left(\frac{\omega}{c}x\right)$$

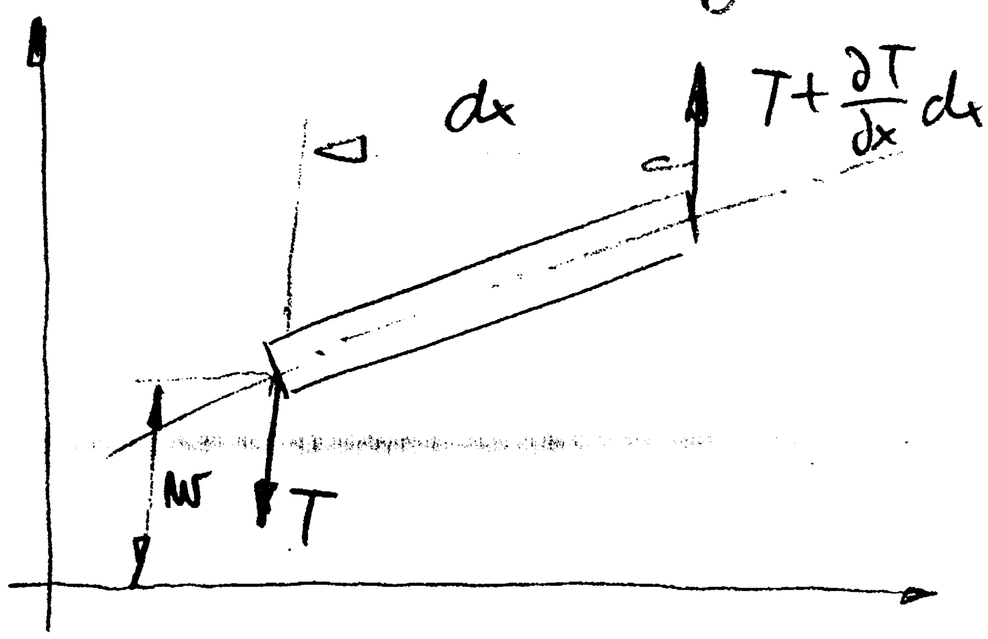
$$D\cdot\frac{\omega}{c}\cos\left(\frac{\omega}{c}l\right) = 0$$

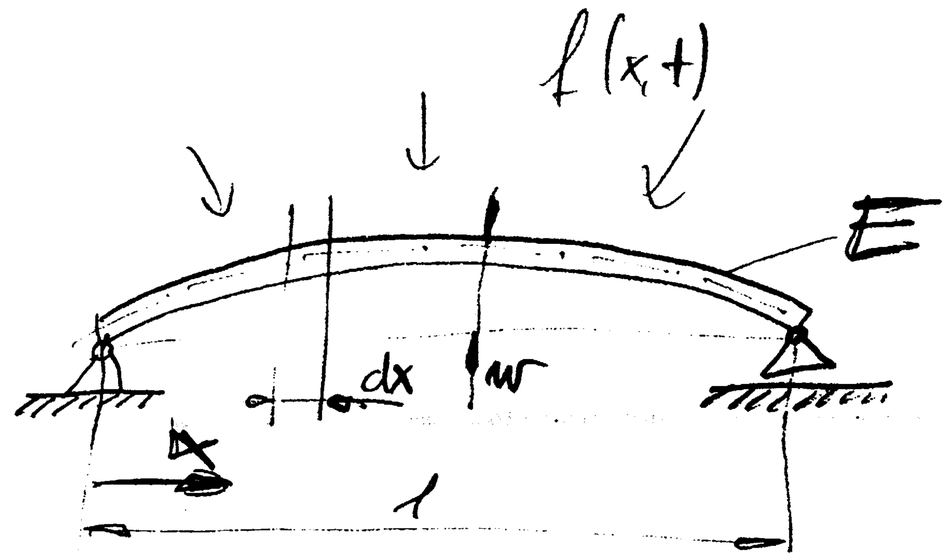
$$\frac{\omega}{c}l = 0 = \left(n - \frac{1}{2}\right)\pi$$



RAVNINSKA UPOGIBNA NIHANJA

Euler-Bernoully-jeva teorija





PREDPOSTAVKE:
- mali poti
- planost prerezov
- zanemarimo strižne deformacije
- zanemarimo rotacijsko vztrajnost

T - prečna sila

$\eta = \frac{m}{l}$ - masa na dolžino

$I(x)$ - vztrajnostni moment preseka $[m^4]$

y smer: $\sum F_{iy} = m\ddot{y}$

$$-T + T\frac{\partial T}{\partial x}dx = n\,dx = \overbrace{\rho A(x)dx}^{dV} \cdot \frac{\partial^2 w}{\partial x^2}$$

$$T = \frac{\partial M}{\partial x} \qquad M = -EI\frac{\partial w^2}{\partial x^2}$$

$$\boxed{\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w}{\partial x^2}\right) + \rho A(x)\frac{\partial^2 w}{\partial t^2} = 0}$$

EULER-BERNOULLI-jeva ENAČBA

Če pa je prerez in vztrajnostni moment konstanten

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0$$

$$\boxed{\frac{EI}{\eta}\cdot\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = 0}$$

EULER-BERNOULLI-jeva ENAČBA za konstanten prerez in za homogen material

METODE	UPOGIBNI MOMENT	STRIŽNA SILA	ROTACIJSKA IZTR.	STRIŽNA DEFOR.
EULER-BERNOULLI-jeva	+	+	—	—
STRIŽNA	+	+	—	+
RAYLEIGH	+	+	+	—
TIMOSHENKO	+	+	+	+

Iščemo $w = w(x,t)$

Ločimo kraj in čas $w(x,t) = X(x)\,T(t)$

$$\frac{\partial^4 w}{\partial x^4} = X^{IV} \cdot T$$

$$\frac{\partial^4 w}{\partial t^2} = X \cdot \ddot{T}$$

$$\frac{EI}{\mu} X^{IV} T + X \cdot \ddot{T} = 0 \quad / X \cdot T \qquad c^2 = \frac{EI}{\mu}$$

$$c^2 \frac{X^{IV}}{X} = -\frac{\ddot{T}}{T} = \text{konst} = \omega^2$$

$$L(x) = D(t)$$

$$\ddot{T} + \omega^2 T = 0$$

$$T(t) = A \cdot \cos \omega t + B \sin \omega t$$

A, B konst. f (začetnih robnih pogojih)

$$X^{IV} - \beta^4 X \qquad \beta^4 = \left(\frac{\omega}{c}\right)^2$$

$$X(x) = C e^{\lambda x}$$

$$X^{IV} = C \lambda^4 \cdot e^{\lambda x}$$

$$C e^{\lambda x} \underbrace{(\lambda^4 - \beta^4)}_{\text{karakteristična enačba}} = 0$$

$$\lambda^4 - \beta^4 = 0$$

$$\lambda_{1,2} = \pm \beta$$

$\lambda_{3,4} = \pm i\beta$ z imaginarna korena

$$X(x) = C_1 e^{\beta x} + C_2 e^{-\beta x} + C_3 e^{i\beta x} + C_4 e^{-i\beta x}$$

$$X(x) = D_1 \operatorname{ch} \beta x + D_2 \operatorname{sh} \beta x + D_3 \cos \beta x + D_4 \sin \beta x$$

$D_1, \dots, D_4$ konstante, f robnih pogojev

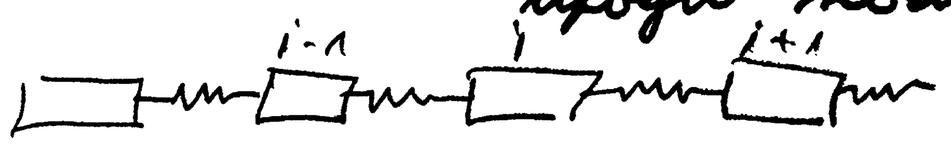
APROKSIMATIVNE METODE

METODE PREVOSNIH MATRIK - MPM METODE

Zaslužna gospoda sta: TOLLE, HOLZER

KJE SE UPORABLJA? Za linearne sisteme. Metoda se uporablja predvsem pri - ~~gr~~ osnih nihanjih palic
- torzijskih nihanj
- upogib nosilcev

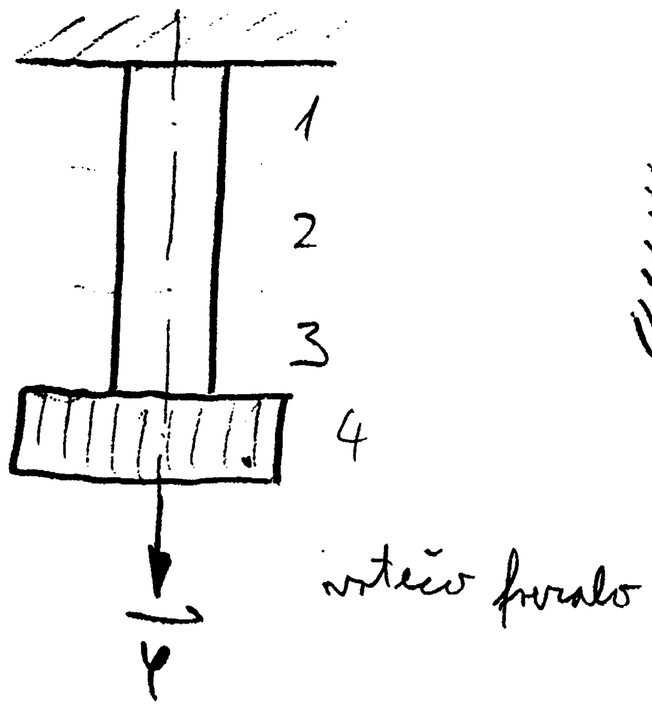
PRIMER:

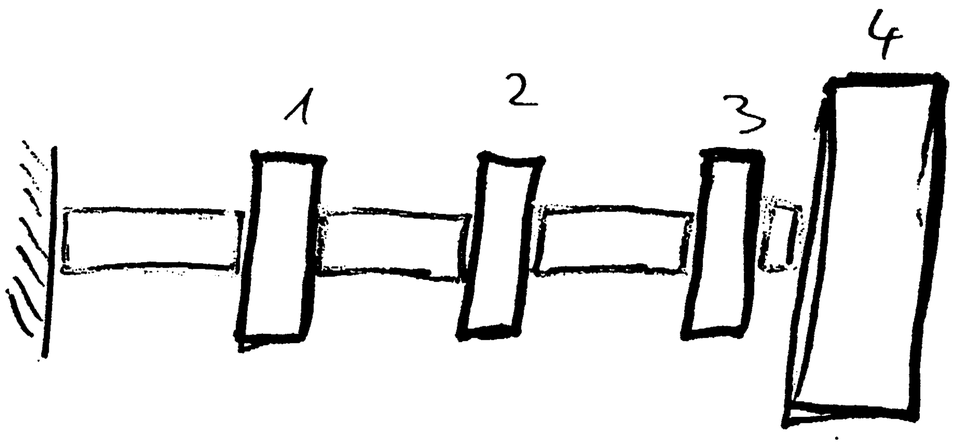


↳ diskretizirani sistem na osno obremenitev

$$[k] = \begin{bmatrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{bmatrix}$$

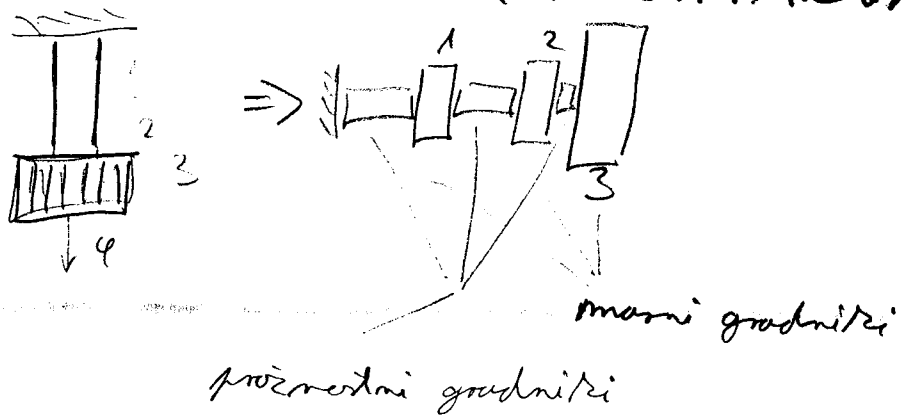
OSNOVNA IDEJA MEHANIKE





OSNOVNA GRADNIKA
- osni gradnik (osni vztrajnost - I_{zz})
(prečnostni element, brez mase)

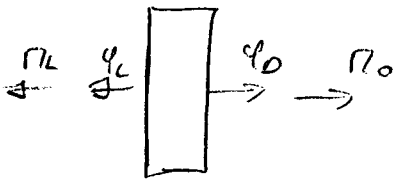
MPM – TORZIJSKA NIHANJA



Vektor stanja $\{V\} = \begin{Bmatrix} \varphi \\ M_t \end{Bmatrix}$ – zasuk
– torzijski moment

Ugotovimo prehod $\{V\}$ preko osnovnih gradnikov

NASOJ ELEMENT



$$\varphi_D - \varphi_L = 0 \Rightarrow \varphi_D = \varphi_L$$
$$M_D - M_L = J \cdot \ddot{\varphi} = -\omega^2 J \cdot \varphi$$

Predpostavimo harmonično nihanje (gibanje)

$\varphi_L = \varphi_L \cdot \sin(\omega t)$

$$\varphi_D = 1 \cdot \varphi_L + 0 \cdot M_L$$
$$M_D = -\omega^2 J \varphi_L + 1 M_L$$

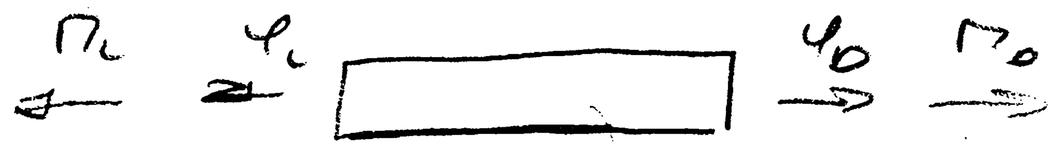
ŽELIMO

$$\{V\}_D = \begin{bmatrix} \text{PREVOSNA} \\ \text{MATRIKA} \end{bmatrix} \cdot \{V\}_L$$

$$\begin{Bmatrix} \varphi \\ M \end{Bmatrix}_D = \begin{bmatrix} 1 & 0 \\ -J\omega^2 & 1 \end{bmatrix} \begin{Bmatrix} \varphi \\ M \end{Bmatrix}_L$$

$[M]$ – prenosna matrika preko mase

PROŽUOSTUI ELEMENT



k_T - torzijska togost

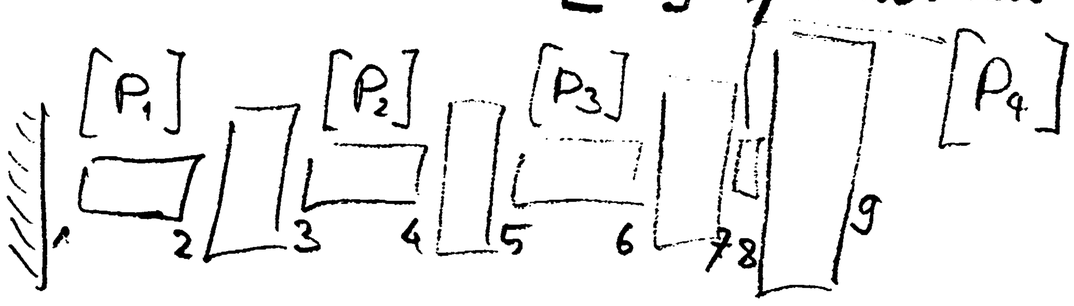
$$M_D - M_L = 0 \qquad M_D = M_L$$

$$\varphi_D - \varphi_L = \frac{M_L}{k_T} \Rightarrow \varphi_D = \varphi_L + \frac{M_L}{k_T}$$

Za torziji $M_t = k_t \cdot \varphi = \frac{G \cdot I_T}{l} \cdot \varphi$

$$\begin{Bmatrix} \varphi \\ M \end{Bmatrix}_D = \begin{bmatrix} 1 & \frac{1}{k_T} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \varphi \\ M \end{Bmatrix}_L$$

[P] prenosna matrika preko polja



$[M_1]$ $[M_2]$ $[M_3]$ $[M_4]$

$$\{V\}_2 = [P_1]\{V_1\}_1$$

$$\{V\}_3 = [M_1]\{V_2\}_2 = [M_1][P_1]\{V_1\}_1$$

$$\vdots$$

$$\{V\}_9 = [M_4]\{V\}_8 = \underbrace{[M_4][P_4][M_3][P_3] \dots [M_1][P_1]}_{[PM]}\{V\}_1$$

ROBNI POGOJI

$$\begin{Bmatrix} \varphi_1 \\ M_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_1 \end{Bmatrix} \qquad \begin{Bmatrix} \varphi_9 \\ M_9 \end{Bmatrix} = \begin{Bmatrix} \varphi_9 \\ M_9 \end{Bmatrix}$$

$$\begin{Bmatrix} \varphi_9 \\ M_9 \end{Bmatrix} = \underbrace{\begin{bmatrix} PM_{11} & PM_{12} \\ PM_{21} & PM_{22} \end{bmatrix}}_{\text{prenosna matrika celotnega sistema}} \cdot \begin{Bmatrix} \varphi_1 \\ M_1 \end{Bmatrix}$$

$$\varphi_9 = PM_{12} \cdot M_1$$

$$\boxed{0 = PM_{22} \cdot M_1}$$

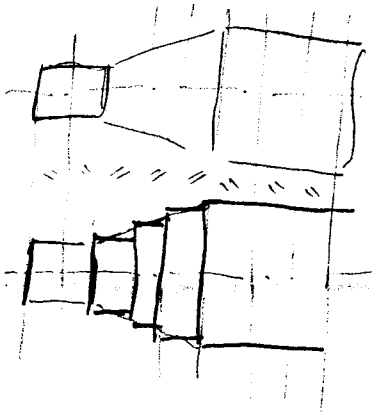
$$PM_{22} = 0 = P(\omega^4) = 0$$

$\omega_{01}, \omega_{02}, \dots$

↳ prenosna matrika preko sistema

z vrednostjo nosil elementov

MPM – UPOGIBNA NIHANJA



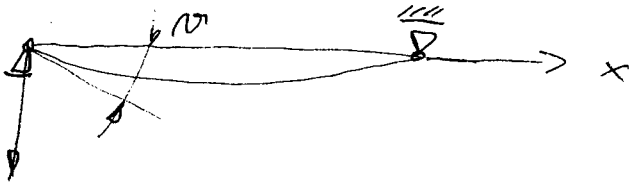
Prevedemo v sistem z vzporednimi prerezi

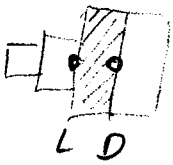
Vektor stanja pri upogibu

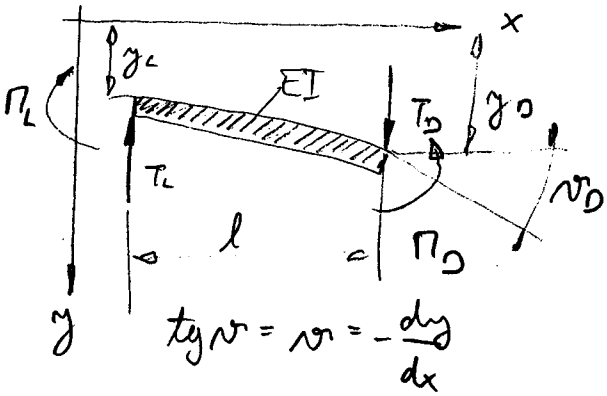
$$\{V\} = \begin{Bmatrix} y \\ v \\ M \\ T \end{Bmatrix}$$

- y – upogibna deformacija (PESTEL, LECKIE)
- v – zavrtek
- M – upogibni moment
- T – prečna sila

PREVOS PREKO BREZMASNEGA ELASTIČNEGA POLJA







$$\operatorname{tg} v = v = -\frac{dy}{dx}$$

$$M = -EI \cdot y''$$

$$T = -EI \cdot y''' = \frac{dM}{dx}$$

$$q = -\frac{dT}{dx} = EI\, y^{IV}$$

$q = 0 \rightarrow$ nov začetek – brezmasni element

$$(EI y'')'' = 0$$

$$y^{IV} = 0 \qquad /4 \times \int$$

$$y(x) = A + Bx + Cx^2 + Dx^3$$

LEVI ROB POMAKNUTO V $x = 0$

$$y(0) = A = y_L$$
$$y'(0) = B = -\gamma' = -\vartheta$$
$$y''(0) = 2C = -\frac{M_L}{EI}$$
$$y'''(0) = 6D = -\frac{T_L}{EI}$$

$$y(x) = y_L - \vartheta_L x - \frac{M_L}{2EI} \cdot x^2 - \frac{T_L}{6EI} x^3$$

SADA GREMO NA DESNI ROB $x = l$

$$y_D = -y_L - \vartheta_L \cdot l + \frac{M_L}{2EI} \cdot l^2 + \frac{T_L}{6EI} l^3$$

$$\vartheta_D = \vartheta_L + \frac{M_L}{EI} l + \frac{T_L}{2EI} l^2$$

$$M_D = M_L + T_L \cdot l$$

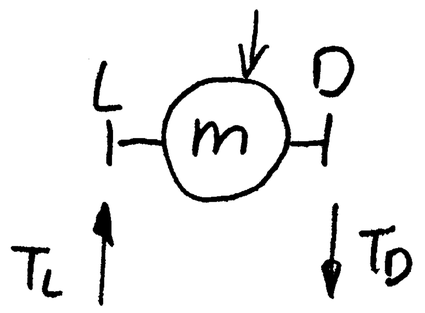
$$T_D = T_L$$

$$\{V\}_D = [P]\{V\}_L$$ ili pišemo $\{V\} = \begin{Bmatrix} -y \\ \vartheta \\ M \\ T \end{Bmatrix}$

$$[P] = \begin{bmatrix} 1 & l & \frac{l^2}{2EI} & \frac{l^3}{6EI} \\ 0 & 1 & \frac{l}{EI} & \frac{l^2}{2EI} \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Prenosna matrica preko homogena polja

PRENOS PREKO TOČKOVNE MASE

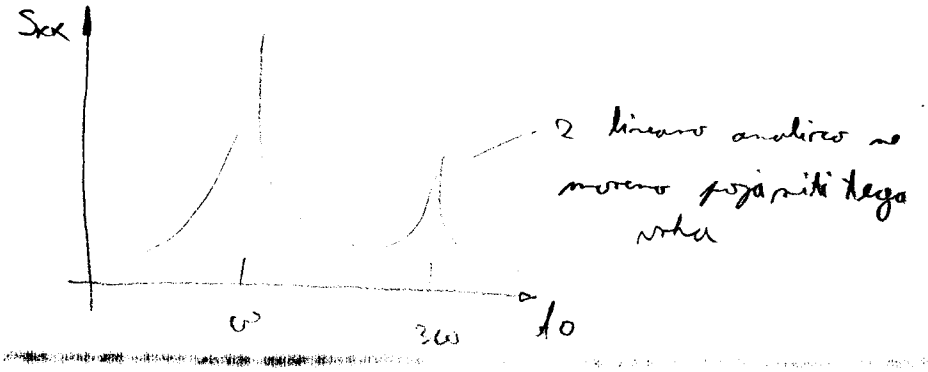


$$T_D - T_L = m\ddot{y} = -m\omega^2 y_L$$
$$T_D = -m\omega^2 y_L + T_L$$
$$\{V\} = [M]\{V\}_L$$

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ m\omega^2 & 0 & 0 & 1 \end{bmatrix}$$

Prenosna matrica preko točkovne mase

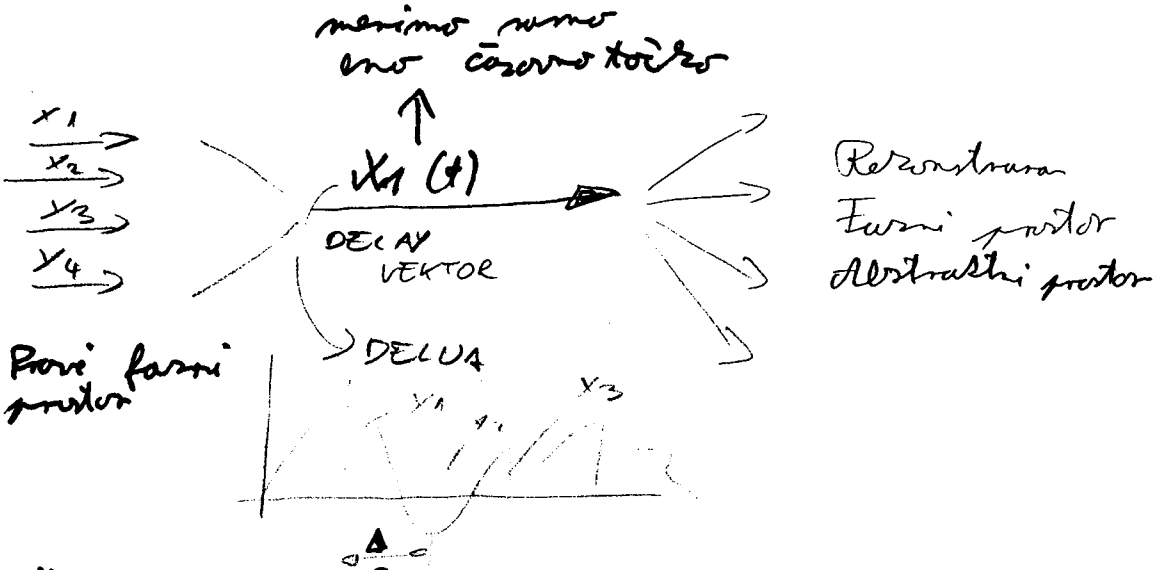
NELINEARNI SISTEMI



VPRAŠAJE ?

	LIN.	NEL	
LAST	ω_0 $\neq f\left(\frac{x_0}{\dot{x}(0)}\right)$	ni samo ena lastna frekvenca ⇒ funkcija začetnih pogojev	$\omega, 3\omega$ $\omega = f\left(\frac{x(0)}{\dot{x}(0)}\right)$

F TAKENS - napiše en članek za velikost prostorskih stopenj



$$\ddot{x}_1 + \omega^2 x_1 = -\frac{A^3}{4}\cos 3\omega t$$

$$x_1(t) = x_{1H} + x_{1P}$$

$$x_{1H} = D_1 \cdot \cos\omega t + D_2 \sin\omega t$$

$$x_{1P} = C \cdot \cos 3\omega t$$

$$(-C\,9\omega^2 + C\omega^2)\cos 3\omega t = -\frac{A^3}{4}\cos 3\omega t$$

$$C = \frac{A^3}{32\omega^2}$$

$$x_1(t) = D_1\cos\omega t + D_2\cos\omega t + \frac{A^3}{32\omega^2}\cos 3\omega t$$

$$x_1(0)=0 = D_1 + \frac{A^3}{32\omega^2} \Rightarrow D_1 = -\frac{A^3}{32\omega^2}$$

$$\dot{x}_1(0)=0 = D_2\omega \rightarrow D_2=0$$

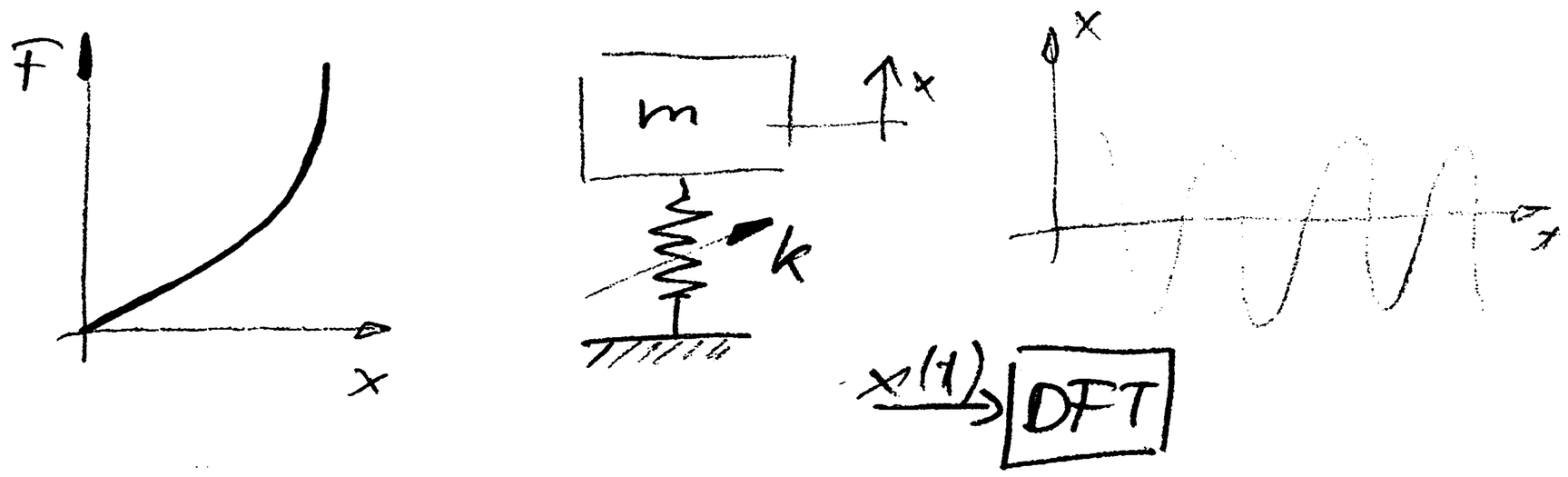
$$x_1(t) = -\frac{A^3}{32}\omega^2 \cos\omega t + \frac{A^3}{32\omega^2}\omega^2 \cos 3\omega t$$

$$\boxed{x(t) = \left(A - \frac{\mu \cdot A^3}{32\omega^2}\right)\cos\omega t + \frac{\mu A^3}{32\omega^2}\cos 3\omega t}$$

→ REŠITEV DUFF, NOVEGA PROBLEMA

BISTVO:

$N=1$, zelo nelinearnost, in v tem imamo 2 frekvenci (mo rezonanci)



$$(\ddot{x}_0 + \mu\ddot{x}_1) + \omega_0^2(x_0 + \mu x_1) + \mu(x_0 + \mu x_1)^3 = 0$$

$$\ddot{x}_0 + \beta\ddot{x}_1 + (\omega_0^2 - \beta\alpha_1)(x_0 + \beta x_1) + \beta(x_0^3 + 3x_0\beta x_1 + 3x_0(\beta x_1)^2 + (\beta x_1)^3) =$$

μ^0: $\ddot{x}_0 + \omega^2 x_0 = 0$

μ^1: $\ddot{x}_1 + \omega^2 x_1 - \alpha_1 x_0 + x_0^3 = 0$

μ^2

$x_0(0) = 0$

$x_1(0) = 0$

$$x_0(t) = A\cdot\cos\omega t + B\sin\omega t$$

$x_0(t) = A\cos\omega t$ → generiram vzbu

$$\ddot{x}_1 + \omega^2 x_1 = \alpha_1 x_0 - x_0^3 = \alpha_1 A\cos\omega t - A^3\cos^3\omega t$$

$$\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$$

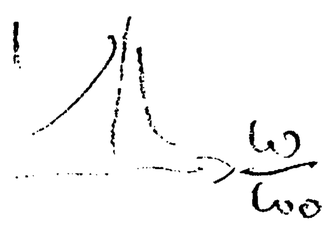
$$\ddot{x}_1 + \omega^2 x_1 = \alpha_1 A\cos\omega t - A^3\frac{3}{4}\cos\omega t - \frac{A^3}{4}\cos 3\omega t$$

$$\ddot{x}_1 + \omega^2 x_1 = \underbrace{\left(\alpha_1\cdot A - \frac{3}{4}A^3\right)}_{x_1(t) = k\cdot t\cos\omega t}\cos\omega t - A^3\frac{3}{4}\cos 3\omega t$$

ZA ONEMOGOČI GIBANJA

$$\alpha_1 A - \frac{3}{4}A^3 = 0$$

$$\alpha_1 = \frac{3}{4}A^2$$



OSNOVNA IDEJA !

Z razliko od lin. nihanj je perioda funkcija poč. pogojev.

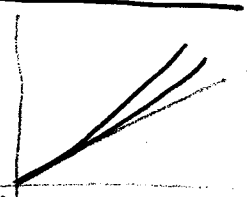
$\left.\begin{matrix}\omega_0\\ T\end{matrix}\right\} f\left(\begin{matrix}\text{poč.}\\ \text{pogojev}\end{matrix}\right)$

APROKSIMATIVNE METODE

Vostra nihanja $\omega = 1$

LINDSTEDT-OVA $\underbrace{\text{PRETURBACNSKA}}_{\text{male nelinearnosti}}$ METODA

METODA MALIH MOTENJ



– KJE JE MEJA

OSNOVNA ENAČBA

$$\ddot{x} + \omega_0^2 x + \mu x^3 = 0$$ DUFFING-OVA ENAČBA (vzmetenje)

μ – mali parameter nelinearnosti

↳ iščemo v bližini linearnih vrednosti (rešitev)

ZAČETNI POGOJI $x(0) = A$

$\dot{x}(0) = 0$

Nekoč vrsta po μ-ju

$$x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \dots$$

$$\omega^2 = \omega_0^2 + \mu\alpha_1 + \mu^2\alpha_2$$ ZANEMARIMO

α_1, α_2 – funkcije amplitude

↳ $\omega_0^2 = \omega^2 - \mu\alpha_1$