Matematika 2

Dvojni in trojni integrali

(1) Izračunaj integrala:

(a) $\displaystyle\int_1^2 dx \int_0^1 (x+y)^2\,dy$,

(b) $\displaystyle\int_0^1 dy \int_1^2 (x+y)^2\,dx$.

Rešitev: (a) Računajmo:

$$\begin{aligned}
\int_1^2 dx \int_0^1 (x+y)^2\,dy &= \int_1^2 dx \int_0^1 (x^2+2xy+y^2)\,dy,\\
&= \int_1^2 dx \left(x^2y + xy^2 + \frac{y^3}{3}\right)\Bigg|_{y=0}^{y=1},\\
&= \int_1^2 dx \left(x^2 + x + \frac{1}{3}\right),\\
&= \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{3}\right)\Bigg|_{x=1}^{x=2},\\
&= \left(\frac{8}{3} + \frac{4}{2} + \frac{2}{3}\right) - \left(\frac{1}{3} + \frac{1}{2} + \frac{1}{3}\right),\\
&= \frac{8}{3} + \frac{3}{2},\\
&= \frac{25}{6}.
\end{aligned}$$

(b) Sedaj je:

$$\begin{aligned}
\int_0^1 dy \int_1^2 (x+y)^2\,dx &= \int_0^1 dy \int_1^2 (x^2+2xy+y^2)\,dx,\\
&= \int_0^1 dy \left(\frac{x^3}{3} + x^2y + xy^2\right)\Bigg|_{x=1}^{x=2},\\
&= \int_0^1 dy \left(\left(\frac{8}{3} + 4y + 2y^2\right) - \left(\frac{1}{3} + y + y^2\right)\right),\\
&= \int_0^1 dy \left(y^2 + 3y + \frac{7}{3}\right),\\
&= \left(\frac{y^3}{3} + \frac{3y^2}{2} + \frac{7y}{3}\right)\Bigg|_{y=0}^{y=1},\\
&= \frac{1}{3} + \frac{3}{2} + \frac{7}{3},\\
&= \frac{25}{6}.
\end{aligned}$$

Vidimo, da sta oba integrala enaka. V splošnem lahko zamenjamo vrstni red integracije, kadar integriramo po pravokotniku. □

(2) Izračunaj integrale:

(a) $\displaystyle\int_0^2 dy \int_0^1 (x^2 + 2y)\, dx,$

(b) $\displaystyle\int_0^1 dy \int_1^3 \frac{x+y}{1+y^2}\, dx,$

(c) $\displaystyle\int_0^1 dx \int_x^1 (x + y)\, dy,$

(d) $\displaystyle\int_1^2 dx \int_{1/x}^x \frac{x^2}{y^2}\, dy,$

(e) $\displaystyle\int_0^{2\pi} d\phi \int_{\sin\phi}^1 r\, dr.$

Rešitev:

(a)
$$\begin{aligned}\int_0^2 dy \int_0^1 (x^2 + 2y)\, dx &= \int_0^2 dy \left(\frac{x^3}{3} + 2xy\right)\Bigg|_{x=0}^{x=1} = \int_0^2 dy \left(\frac{1}{3} + 2y\right),\\ &= \left(\frac{y}{3} + y^2\right)\Bigg|_{y=0}^{y=2} = \frac{2}{3} + 4,\\ &= \frac{14}{3}.\end{aligned}$$

(b)
$$\begin{aligned}\int_0^1 dy \int_1^3 \frac{x+y}{1+y^2}\, dx &= \int_0^1 dy \left(\frac{x^2}{2(1+y^2)} + \frac{xy}{1+y^2}\right)\Bigg|_{x=1}^{x=3},\\ &= \int_0^1 dy \left(\left(\frac{9}{2(1+y^2)} + \frac{3y}{1+y^2}\right) - \left(\frac{1}{2(1+y^2)} + \frac{y}{1+y^2}\right)\right),\\ &= \int_0^1 dy \left(\frac{4}{1+y^2} + \frac{2y}{1+y^2}\right) = \left(4 \operatorname{arc\,tg} y + \ln(1+y^2)\right)\Big|_{y=0}^{y=1},\\ &= (4 \operatorname{arc\,tg} 1 + \ln 2) - (4 \operatorname{arc\,tg} 0 + \ln 1),\\ &= \pi + \ln 2.\end{aligned}$$

(c)
$$\begin{aligned}\int_0^1 dx \int_x^1 (x + y)\, dy &= \int_0^1 dx \left(xy + \frac{y^2}{2}\right)\Bigg|_{y=x}^{y=1} = \int_0^1 dx \left(\left(x + \frac{1}{2}\right) - \left(x^2 + \frac{x^2}{2}\right)\right),\\ &= \int_0^1 dx \left(-\frac{3x^2}{2} + x + \frac{1}{2}\right) = \left(-\frac{x^3}{2} + \frac{x^2}{2} + \frac{x}{2}\right)\Bigg|_{x=0}^{x=1},\\ &= -\frac{1}{2} + \frac{1}{2} + \frac{1}{2},\\ &= \frac{1}{2}.\end{aligned}$$

$$\text{(d)} \int_1^2 dx \int_{1/x}^{x} \frac{x^2}{y^2}\, dy = \int_1^2 dx \left(-\frac{x^2}{y}\right)\Bigg|_{y=\frac{1}{x}}^{y=x} = \int_1^2 dx \left(-x + x^3\right),$$

$$= \left(-\frac{x^2}{2} + \frac{x^4}{4}\right)\Bigg|_{x=1}^{x=2} = (-2+4) - \left(-\frac{1}{2} + \frac{1}{4}\right),$$

$$= \frac{9}{4}.$$

$$\text{(e)} \int_0^{2\pi} d\phi \int_{\sin\phi}^{1} r\, dr = \int_0^{2\pi} d\phi \left(\frac{r^2}{2}\right)\Bigg|_{r=\sin\phi}^{r=1} = \int_0^{2\pi} d\phi \left(\frac{1-\sin^2\phi}{2}\right) = \int_0^{2\pi} d\phi \left(\frac{\cos^2\phi}{2}\right),$$

$$= 4\int_0^{\frac{\pi}{2}} d\phi \left(\frac{\cos^2\phi}{2}\right) = 2\int_0^{\frac{\pi}{2}} \cos^2\phi\, d\phi,$$

$$= 2\cdot\frac{1}{2}B\left(\frac{3}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)},$$

$$= \frac{\pi}{2}.$$

□

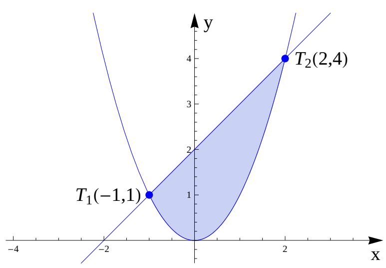
(3) Določi meje integracije in nato zapiši integrale, ki določajo ploščino likov, ki so omejeni z naslednjimi krivuljami:

(a) $y = x^2$ in $y = x + 2$,

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$,

(c) $y = \sqrt{x}$, $y = 6 - x$ in $y = 0$.

Rešitev: (a) Lik je omejen s premico in parabolo.



Presečišči dobimo iz enačbe $x^2 = x + 2$, ki ima rešitvi $x_1 = -1$ in $x_2 = 2$. Pripadajoči točki sta $T_1(-1,1)$ in $T_2 = (2,4)$. Če hočemo lik parametrizirati, bo torej veljalo

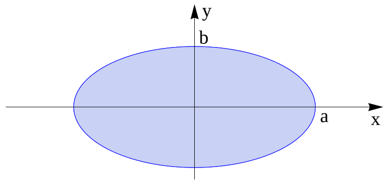
$$x \in [-1, 2],$$
$$y \in [x^2, x+2].$$

Ploščina lika je enaka integralu

$$S = \int_{-1}^{2} dx \int_{x^2}^{x+2} dy.$$

Če bi šli ta integral izračunati, bi dobili $S = \frac{9}{2}$.

(b) Lik je sedaj omejen z elipso $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, ki ima polosi a in b.



Zgornji in spodnji rob elipse sestavljata krivulji $y = \pm\sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}$, od koder sledi

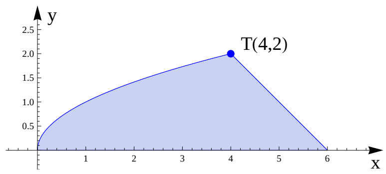
$$x \in [-a, a],$$
$$y \in \left[-\sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}, \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}\right].$$

Ploščina elipse je enaka integralu

$$S = \int_{-a}^{a} dx \int_{-\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}}^{\sqrt{b^2\left(1-\frac{x^2}{a^2}\right)}} dy,$$

ki je enak $S = \pi ab$.

(c) V tem primeru je lik omejen s parabolo in z dvema premicama.



Presečišče premic je v točki $T_1(6, 0)$, presečišče premice $y = 0$ in parabole v točki $T_2(0, 0)$, presečišče premice $y = 6 - x$ in parabole pa lahko dobimo iz naslednje enačbe:

$$6 - x = \sqrt{x},$$
$$36 - 12x + x^2 = x,$$
$$x^2 - 13x + 36 = 0,$$
$$(x - 9)(x - 4) = 0.$$

Če preverimo obe rešitvi, vidimo, da prvotni enačbi ustreza samo rešitev $x = 4$. Presečišče je torej $T_3 = (4, 2)$. Naš lik je od zgoraj omejen z dvema krivuljama, zato ga je lažje parametrizirati v obratnem vrstnem redu kot v prejšnjih primerih. Desni rob lika je namreč krivulja $y = 6 - x$ oziroma $x = 6 - y$, levi rob pa $y = \sqrt{x}$ oziroma $x = y^2$. Parametrizacija se sedaj glasi

$$y \in [0, 2],$$
$$x \in [y^2, 6 - y].$$

Ploščina lika je

$$S = \int_0^2 dy \int_{y^2}^{6-y} dx,$$

kar je enako $S = \frac{22}{3}$. □

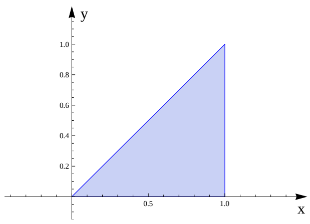
(4) Določi območja, katerih ploščina je podana z naslednjimi integrali:

(a) $\displaystyle\int_0^1 dx \int_0^x dy,$

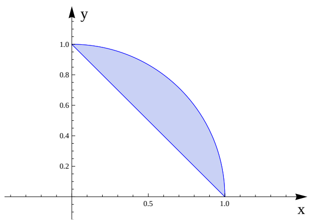
(b) $\displaystyle\int_0^1 dy \int_{1-y}^{\sqrt{1-y^2}} dx,$

(c) $\displaystyle\int_0^1 dx \int_x^{\sqrt{2-x^2}} dy.$

Rešitev: (a) Projekcija lika na abscisno os je interval $[0, 1]$. Na tem intervalu je lik omejen od zgoraj s premico $y = x$ od spodaj pa s premico $y = 0$. Dobimo trikotnik, ki je omejen s premicami $y = 0$, $y = x$ in $x = 1$.



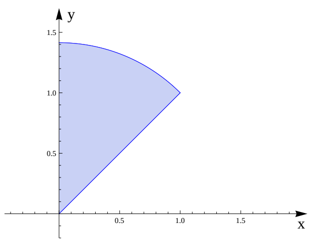
(b) Sedaj je projekcija lika na ordinatno os enaka intervalu $[0, 1]$. Levi rob lika je premica $x = 1 - y$ oziroma $y = 1 - x$, desni rob pa krivulja $x = \sqrt{1 - y^2}$. To enačbo lahko s kvadriranjem preoblikujemo v enačbo $x^2 + y^2 = 1$, kar pomeni, da gre za del krožnice s polmerom $R = 1$, ki leži v prvem kvadrantu. Naš lik je torej krožni odsek.



(c) Lik je sedaj od zgoraj omejen s krožnico $y = \sqrt{2 - x^2}$ s polmerom $R = \sqrt{2}$ od spodaj pa s premico $y = x$. Njuno presečišče je določeno z enačbo:

$$\begin{aligned} x &= \sqrt{2 - x^2}, \\ x^2 &= 2 - x^2, \\ x^2 &= 1. \end{aligned}$$

Zanima nas samo točka, ki leži v prvem kvadrantu, kar pomeni, da je $x = 1$, presečišče pa $T(1, 1)$. Lik je v tem primeru krožni izsek.



$\square$

(5) Izračunaj integrale:

(a) funkcije $f(x,y) = xy$ po trikotniku, ki ga omejujejo premice $y = 0$, $y = x$ in $x = 1$,
(b) funkcije $f(x,y) = x^2 + xy^3$ po kvadratu $0 \le x \le 1$, $1 \le y \le 2$,
(c) funkcije $f(x,y) = x^2 y$ po liku med krivuljama $y = x$ in $y = x^2$.

Rešitev: (a) Trikotnik, ki ga omejujejo premice $y = 0$, $y = x$ in $x = 1$, parametriziramo z $x \in [0,1]$ in $y \in [0,x]$. Sledi

$$\int_0^1 dx \int_0^x xy\,dy = \int_0^1 dx \left(\frac{xy^2}{2}\right)\Bigg|_{y=0}^{y=x} = \int_0^1 dx \left(\frac{x^3}{2}\right) = \frac{x^4}{8}\Bigg|_{x=0}^{x=1} = \frac{1}{8}.$$

(b) Sedaj bomo integrirali po območju $x \in [0,1]$ in $y \in [1,2]$, kar nam da:

$$\begin{aligned}\int_0^1 dx \int_1^2 (x^2 + xy^3)\,dy &= \int_0^1 dx \left(x^2 y + \frac{xy^4}{4}\right)\Bigg|_{y=1}^{y=2} = \int_0^1 dx \left((2x^2 + 4x) - \left(x^2 + \frac{x}{4}\right)\right), \\ &= \int_0^1 dx \left(x^2 + \frac{15x}{4}\right) = \left(\frac{x^3}{3} + \frac{15x^2}{8}\right)\Bigg|_{x=0}^{x=1} = \frac{1}{3} + \frac{15}{8}, \\ &= \frac{53}{24}.\end{aligned}$$

(c) Območje med parabolo in premico lahko parametriziramo z $x \in [0,1]$ in $y \in [x^2, x]$, kar nam da:

$$\begin{aligned}\int_0^1 dx \int_{x^2}^x x^2 y\,dy &= \int_0^1 dx \left(\frac{x^2 y^2}{2}\right)\Bigg|_{y=x^2}^{y=x} = \int_0^1 dx \left(\frac{x^4}{2} - \frac{x^6}{2}\right) = \left(\frac{x^5}{10} - \frac{x^7}{14}\right)\Bigg|_{x=0}^{x=1}, \\ &= \frac{1}{10} - \frac{1}{14} = \frac{1}{35}.\end{aligned}$$

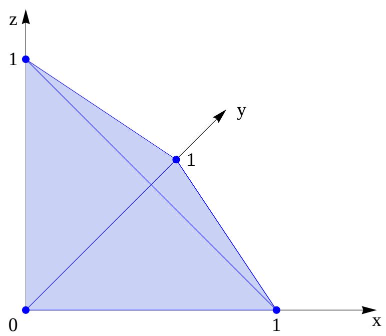
□

(6) Izračunaj volumen telesa, ki leži pod grafom funkcije $f(x,y) = 1 - x - y$ nad trikotnikom, ki ga omejujejo premice $x = 0$, $y = 0$ in $y = 1 - x$.

Rešitev: Volumen telesa, ki leži pod grafom funkcije dveh spremenljivk f nad likom L, je enak

$$V = \iint_L f(x,y)\,dx\,dy.$$

Naše telo ima obliko štiristrane piramide.



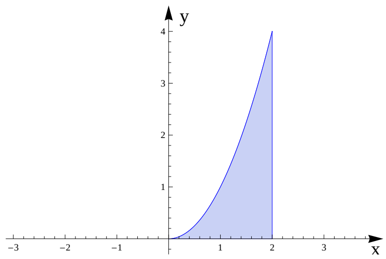
Njen volumen je enak:

$$\begin{aligned}
V = \int_0^1 dx \int_0^{1-x} (1-x-y)\,dy &= \int_0^1 dx\, \left(y - xy - \frac{y^2}{2} \right) \Bigg|_{y=0}^{y=1-x}, \\
&= \int_0^1 dx\, \left((1-x) - x(1-x) - \frac{(1-x)^2}{2} \right), \\
&= \int_0^1 dx\, \left(1 - x - x + x^2 - \frac{1-2x+x^2}{2} \right), \\
&= \int_0^1 dx\, \left(\frac{x^2}{2} - x + \frac{1}{2} \right), \\
&= \left(\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{2} \right) \Bigg|_{x=0}^{x=1} = \frac{1}{6} - \frac{1}{2} + \frac{1}{2}, \\
&= \frac{1}{6}.
\end{aligned}$$

□

(7) Izračunaj središče lika, ki je omejen s krivuljami $y = x^2$, $y = 0$ in $x = 2$.

Rešitev: Iščemo središče lika, ki ima za zgornji rob kvadratno parabolo.



Središče ravninskega lika L lahko izračunamo s pomočjo formul:

$$x_* = \frac{\iint_L x\,dx\,dy}{S},$$
$$y_* = \frac{\iint_L y\,dx\,dy}{S},$$

kjer je S ploščina lika L, ki je v našem primeru enaka

$$S = \int_0^2 x^2\,dx = \frac{x^3}{3}\Bigg|_0^2 = \frac{8}{3}.$$

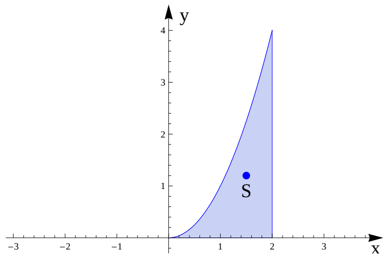
Sedaj bomo izračunali oba integrala v števcih zgornjih ulomkov. Naš lik lahko parametriziramo s predpisoma $x \in [0,2]$ in $y \in [0, x^2]$, od koder dobimo:

$$\iint_L x\,dx\,dy = \int_0^2 dx \int_0^{x^2} x\,dy = \int_0^2 dx\,\,(xy)\Big|_{y=0}^{y=x^2} = \int_0^2 x^3\,dx = \frac{x^4}{4}\Bigg|_{x=0}^{x=2} = 4,$$
$$\iint_L y\,dx\,dy = \int_0^2 dx \int_0^{x^2} y\,dy = \int_0^2 dx\,\left(\frac{y^2}{2}\right)\Bigg|_{y=0}^{y=x^2} = \int_0^2 \frac{x^4}{2}\,dx = \frac{x^5}{10}\Bigg|_{x=0}^{x=2} = \frac{32}{10}.$$

Koordinati središča lika sta torej:

$$x_* = \frac{\iint_L x\,dx\,dy}{S} = \frac{4}{\frac{8}{3}} = \frac{3}{2},$$
$$y_* = \frac{\iint_L y\,dx\,dy}{S} = \frac{\frac{32}{10}}{\frac{8}{3}} = \frac{6}{5}.$$

Poglejmo še skico.



□

(8) S prevedbo na polarne koordinate izračunaj integrale:

(a) $\displaystyle\iint_D e^{-(x^2+y^2)}\,dx\,dy$, kjer je D krog $x^2+y^2 \leq 1$,

(b) $\displaystyle\iint_D \frac{dx\,dy}{1-x^2-y^2}$, kjer je D krog $x^2+y^2 \leq 1/2$,

(c) $\displaystyle\iint_D \sqrt{x^2+y^2}\,dx\,dy$, kjer je D kolobar $4 \leq x^2+y^2 \leq 9$.

Rešitev: Kadar integriramo po krogih ali kolobarjih, je namesto v kartezičnih lažje računati v polarnih koordinatah. Pri tem uporabljamo substitucijo:

$$x = r\cos\phi,$$
$$y = r\sin\phi.$$

Izraz $dx\,dy$ moramo v tem primeru zamenjati z izrazom $r\,dr\,d\phi$. Velja si zapomniti zvezi:

$$x^2+y^2 = r^2,$$
$$\sqrt{x^2+y^2} = r.$$

(a) Najprej bomo izračunali integral $\iint_D e^{-(x^2+y^2)}\,dx\,dy$ po krogu $x^2+y^2 \leq 1$. V polarnih koordinatah lahko ta integral parametriziramo s predpisoma $\phi \in [0, 2\pi]$ in $r \in [0,1]$. Sledi

$$\iint_D e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi} d\phi \int_0^1 e^{-r^2} r\,dr.$$

Integral na desni lahko izračunamo z uvedbo nove spremenljivke $-r^2 = t$, ki nam da $-2r\,dr = dt$ in

$$\int e^{-r^2} r\,dr = -\frac{1}{2}\int e^t\,dt = -\frac{1}{2}e^t + C = -\frac{1}{2}e^{-r^2} + C.$$

Od tod dobimo

$$
\begin{aligned}
\iint_D e^{-(x^2+y^2)}\,dx\,dy &= \int_0^{2\pi} d\phi \int_0^1 e^{-r^2} r\,dr = \int_0^{2\pi} d\phi \left(-\frac{1}{2}e^{-r^2}\right)\Big|_{r=0}^{r=1}, \\
&= \int_0^{2\pi} d\phi \left(-\frac{1}{2}e^{-1}+\frac{1}{2}\right) = \frac{1}{2}\left(1-e^{-1}\right)\int_0^{2\pi} d\phi = \frac{1}{2}\left(1-e^{-1}\right)\phi\Big|_{\phi=0}^{\phi=2\pi}, \\
&= \frac{1}{2}\left(1-e^{-1}\right)\cdot 2\pi = \pi\left(1-e^{-1}\right).
\end{aligned}
$$

(b) Sedaj bomo integrirali funkcijo $f(x,y) = \frac{1}{1-x^2-y^2} = \frac{1}{1-r^2}$ po krogu s polmerom $R = \frac{\sqrt{2}}{2}$. V polarnih koordinatah ga parametriziramo s predpisoma $\phi \in [0, 2\pi]$ in $r \in \left[0, \frac{\sqrt{2}}{2}\right]$. Sledi

$$
\iint_D \frac{dx\,dy}{1-x^2-y^2} = \int_0^{2\pi} d\phi \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{1-r^2} r\,dr.
$$

Uvedimo novo spremenljivko $1-r^2 = t$, ki nam da $-2r\,dr = dt$ in

$$
\int \frac{1}{1-r^2} r\,dr = -\frac{1}{2}\int \frac{dt}{t} = -\frac{1}{2}\ln t + C = -\frac{1}{2}\ln(1-r^2) + C.
$$

Od tod dobimo

$$
\begin{aligned}
\iint_D \frac{dx\,dy}{1-x^2-y^2} &= \int_0^{2\pi} d\phi \int_0^{\frac{\sqrt{2}}{2}} \frac{1}{1-r^2} r\,dr = \int_0^{2\pi} d\phi \left(-\frac{1}{2}\ln(1-r^2)\right)\Big|_{r=0}^{r=\frac{\sqrt{2}}{2}}, \\
&= \int_0^{2\pi} d\phi \left(-\frac{1}{2}\ln\left(1-\frac{1}{2}\right)\right) = -\frac{1}{2}\ln\left(\frac{1}{2}\right)\int_0^{2\pi} d\phi = \frac{1}{2}\ln 2\cdot\phi\Big|_{\phi=0}^{\phi=2\pi}, \\
&= \pi\ln 2.
\end{aligned}
$$

(c) V tem primeru integriramo po kolobarju $4 \leq x^2+y^2 \leq 9$, ki ga lahko parametriziramo s predpisoma $\phi \in [0, 2\pi]$ in $r \in [2, 3]$. Torej je

$$
\begin{aligned}
\iint_D \sqrt{x^2+y^2}\,dx\,dy &= \int_0^{2\pi} d\phi \int_2^3 r\,r\,dr = \int_0^{2\pi} d\phi \left(\frac{r^3}{3}\right)\Big|_{r=2}^{r=3}, \\
&= \int_0^{2\pi} d\phi \left(\frac{27}{3}-\frac{8}{3}\right) = \frac{19}{3}\int_0^{2\pi} d\phi = \frac{19}{3}\phi\Big|_{\phi=0}^{\phi=2\pi}, \\
&= \frac{38\pi}{3}.
\end{aligned}
$$

□

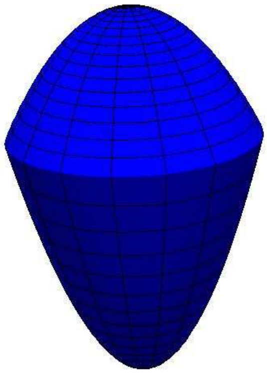
(9) Izračunaj volumen telesa, ki ga omejujeta ploskvi $z = 3-x^2-y^2$ in $z = 2x^2+2y^2$.

Rešitev: Najprej izračunajmo presečišče danih ploskev. Veljati mora:

$$
\begin{aligned}
2x^2+2y^2 &= 3-x^2-y^2, \\
x^2+y^2 &= 1.
\end{aligned}
$$

Tloris telesa je torej omejen s krožnico $x^2+y^2 = 1$. Zgornja ploskev se ujema z grafom funkcije $f(x,y) = 3-x^2-y^2$, spodnja pa z grafom funkcije $g(x,y) = 2x^2+2y^2$.

Dobljeno telo ima naslednjo obliko.



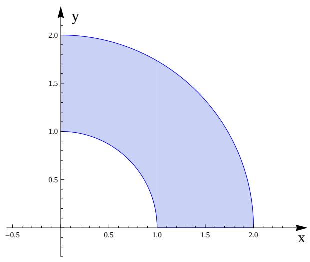
Ker integriramo po krogu s polmerom $R = 1$, bomo raje računali v polarnih koordinatah. Iz enakosti $3 - x^2 - y^2 = 3 - r^2$ in $2x^2 + 2y^2 = 2r^2$ dobimo:

$$
\begin{aligned}
V = \iint_K (f(x,y) - g(x,y))\, dx\, dy &= \int_0^{2\pi} d\phi \int_0^1 (3 - r^2 - 2r^2) r\, dr, \\
&= \int_0^{2\pi} d\phi \int_0^1 (3r - 3r^3)\, dr = \int_0^{2\pi} d\phi \left(\frac{3r^2}{2} - \frac{3r^4}{4} \right) \Bigg|_{r=0}^{r=1}, \\
&= \int_0^{2\pi} d\phi \left(\frac{3}{2} - \frac{3}{4} \right) = \frac{3}{4} \int_0^{2\pi} d\phi, \\
&= \frac{3\pi}{2}.
\end{aligned}
$$

□

(10) Izračunaj središče kolobarja $1 \leq x^2 + y^2 \leq 4$, $x \geq 0$, $y \geq 0$.

Rešitev: Naš lik je del kolobarja, ki leži v prvem kvadrantu.



Lik bomo parametrizirali v polarnih koordinatah s predpisoma $\phi \in \left[0, \frac{\pi}{2}\right]$ in $r \in [1, 2]$. Njegova ploščina je enaka četrtini razlike ploščin večjega in manjšega kroga, kar je enako $S = \frac{1}{4}\left(\pi \cdot 2^2 - \pi \cdot 1^2\right) = \frac{3\pi}{4}$. Poleg tega je:

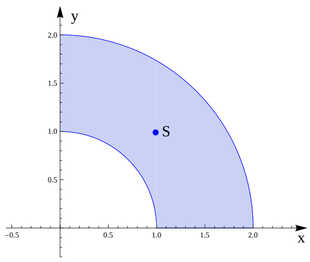
$$
\begin{aligned}
\iint_L x\, dx\, dy = \int_0^{\frac{\pi}{2}} d\phi \int_1^2 r \cos\phi\, r\, dr &= \int_0^{\frac{\pi}{2}} d\phi \left(\cos\phi\, \frac{r^3}{3} \right) \Bigg|_{r=1}^{r=2}, \\
&= \int_0^{\frac{\pi}{2}} d\phi \left(\frac{7}{3} \cos\phi \right) = \frac{7}{3} \int_0^{\frac{\pi}{2}} \cos\phi\, d\phi = \frac{7}{3} \sin\phi \Bigg|_{\phi=0}^{\phi=\frac{\pi}{2}}, \\
&= \frac{7}{3},
\end{aligned}
$$

$$\iint_L y\,dx\,dy = \int_0^{\frac{\pi}{2}} d\phi \int_1^2 r\sin\phi\, r\,dr = \int_0^{\frac{\pi}{2}} d\phi \left(\sin\phi\,\frac{r^3}{3}\right)\Bigg|_{r=1}^{r=2},$$
$$= \int_0^{\frac{\pi}{2}} d\phi \left(\frac{7}{3}\sin\phi\right) = \frac{7}{3}\int_0^{\frac{\pi}{2}} \sin\phi\,d\phi = -\frac{7}{3}\cos\phi\Bigg|_{\phi=0}^{\phi=\frac{\pi}{2}},$$
$$= \frac{7}{3},$$

Od tod dobimo koordinati središča lika:

$$x_* = \frac{\iint_L x\,dx\,dy}{S} = \frac{\frac{7}{3}}{\frac{3\pi}{4}} = \frac{28}{9\pi},$$
$$y_* = \frac{\iint_L y\,dx\,dy}{S} = \frac{\frac{7}{3}}{\frac{3\pi}{4}} = \frac{28}{9\pi}.$$

Poglejmo še skico.



□

(11) Izračunaj težišče kroga $x^2 + y^2 \le 1$, katerega desna polovica je dvakrat gostejša kot leva polovica.

Rešitev: V primeru homogenih likov oziroma teles središče in pa težišče sovpadata. V našem primeru pa je desna polovica kroga težja od leve polovice, zato pričakujemo, da bo težišče malce desno od središča. V splošnem lahko težišče lika L, katerega gostota je dana s funkcijo ρ, izračunamo s pomočjo formul:

$$x_* = \frac{\iint_L x\rho(x,y)\,dx\,dy}{m},$$
$$y_* = \frac{\iint_L y\rho(x,y)\,dx\,dy}{m},$$

kjer je m masa lika. Označimo z ρ_L in ρ_D gostoti leve oziroma desne polovice kroga. Po predpostavki je $\rho_D = 2\rho_L$. Ker sta ploščini obeh polovic kroga enaki $\frac{\pi}{2}$, sta masi leve in desne polovice kroga enaki $m_L = \frac{\rho_L \pi}{2}$ in $m_D = \frac{\rho_D \pi}{2}$. Skupna masa kroga pa je

$$m = m_L + m_D = \frac{\rho_L \pi}{2} + \frac{\rho_D \pi}{2} = \frac{\pi}{2}(\rho_L + \rho_D) = \frac{3\pi}{2}\rho_L.$$

Pri integraciji bomo integracijsko območje razdelili na dva kosa. Levi del parametriziramo s parametroma:

$$\phi \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right],$$
$$r \in [0,1],$$

desni pa s parametroma:

$$\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$
$$r \in [0,1].$$

Od tod dobimo:

$$\begin{aligned}
\iint_L x\rho(x,y)\,dx\,dy &= \iint_{\text{levi}} x\rho_L\,dx\,dy + \iint_{\text{desni}} x\rho_D\,dx\,dy,\\
&= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\phi \int_0^1 r\cos\phi\,\rho_L\,r\,dr + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \int_0^1 r\cos\phi\,\rho_D\,r\,dr,\\
&= \rho_L \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\phi \int_0^1 \cos\phi\, r^2\,dr + \rho_D \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \int_0^1 \cos\phi\, r^2\,dr,\\
&= \rho_L \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d\phi \left(\cos\phi\,\frac{r^3}{3}\right)\Bigg|_{r=0}^{r=1} + \rho_D \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \left(\cos\phi\,\frac{r^3}{3}\right)\Bigg|_{r=0}^{r=1},\\
&= \frac{\rho_L}{3} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\phi\,d\phi + \frac{\rho_D}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\phi\,d\phi,\\
&= \frac{\rho_L}{3} \sin\phi\Big|_{\phi=\frac{\pi}{2}}^{\phi=\frac{3\pi}{2}} + \frac{\rho_D}{3} \sin\phi\Big|_{\phi=-\frac{\pi}{2}}^{\phi=\frac{\pi}{2}},\\
&= \frac{\rho_L}{3}(-1-1) + \frac{\rho_D}{3}(1-(-1)),\\
&= -\frac{2\rho_L}{3} + \frac{2\rho_D}{3},\\
&= \frac{2\rho_L}{3}.
\end{aligned}$$

Od tod dobimo x koordinato težišča

$$x_* = \frac{\iint_L x\rho(x,y)\,dx\,dy}{m} = \frac{\frac{2\rho_L}{3}}{\frac{3\pi}{2}\rho_L} = \frac{4}{9\pi}.$$

Podobno bi lahko izračunali y koordinato težišča. Lažje pa je opaziti, da mora le-ta zaradi simetrije kroga ležati na abscisni osi, od koder sledi $y_* = 0$. □

(12) Izračunaj integrala:

(a) $\displaystyle\int_0^2 dx \int_0^2 dy \int_0^2 x\,dz$,

(b) $\displaystyle\int_0^1 dx \int_0^{1-x} dy \int_0^2 z\,dz$.

Rešitev: (a) Računajmo:

$$\begin{aligned}
\int_0^2 dx \int_0^2 dy \int_0^2 x\,dz &= \int_0^2 dx \int_0^2 dy (xz)\Big|_{z=0}^{z=2} = \int_0^2 dx \int_0^2 2x\,dy,\\
&= \int_0^2 dx\,(2xy)\Big|_{y=0}^{y=2} = \int_0^2 4x\,dx = 2x^2\Big|_{x=0}^{x=2},\\
&= 8.
\end{aligned}$$

(b) Sedaj imamo:

$$\begin{aligned}\int_0^1 dx \int_0^{1-x} dy \int_0^2 z\,dz &= \int_0^1 dx \int_0^{1-x} dy\, \frac{z^2}{2}\Big|_{z=0}^{z=2} = \int_0^1 dx \int_0^{1-x} 2\,dy,\\ &= \int_0^1 dx\,(2y)\Big|_{y=0}^{y=1-x} = \int_0^1 (2-2x)\,dx = (2x-x^2)\Big|_{x=0}^{x=1},\\ &= 1.\end{aligned}$$

□

(13) Določi meje integracije in nato zapiši integrala, ki določata volumna teles, ki sta omejeni z naslednjimi ploskvami:

(a) $x^2 + y^2 = 1$, $z = 0$ in $z = x + 2$,

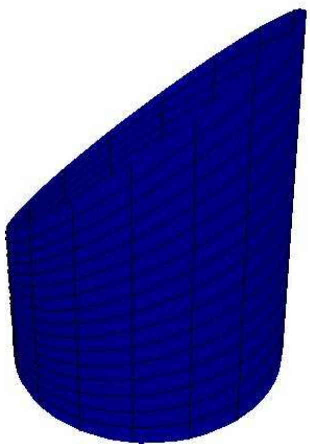
(b) $z = x^2 + y^2$ in $z^2 = x^2 + y^2$.

Rešitev: Pri tej nalogi bomo uporabili cilindrične koordinate, ki so posplošitev polarnih koordinat v tri dimenzije. Med cilindričnimi in kartezičnimi koordinatami imamo naslednjo zvezo:

$$\begin{aligned} x &= r\cos\phi,\\ y &= r\sin\phi,\\ z &= z,\end{aligned}$$

pri integriranju pa moramo izraz $dx\,dy\,dz$ zamenjati z izrazom $r\,dr\,d\phi\,dz$.

(a) Ploskev $x^2 + y^2 = 1$ predstavlja plašč navpičnega valja s polmerom $R = 1$, katerega tloris je krog $x^2 + y^2 \leq 1$. Iskano telo dobimo tako, da ta valj presekamo z vodoravno ravnino $z = 0$ in pa s poševno ravnino $z = x + 2$.



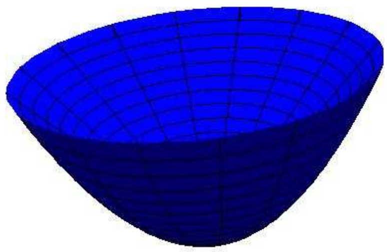
Telo lahko parametriziramo s predpisom:

$$\begin{aligned} \phi &\in [0, 2\pi],\\ r &\in [0, 1],\\ z &\in [0, r\cos\phi + 2].\end{aligned}$$

Volumen telesa je enak

$$V = \int_0^{2\pi} d\phi \int_0^1 dr \int_0^{r\cos\phi+2} r\,dz = \ldots = 2\pi.$$

(b) Sedaj imamo telo, ki je omejeno s plaščem stožca $z = \sqrt{x^2 + y^2}$ in s plaščem paraboloida $z = x^2 + y^2$. Tloris telesa je spet krog $x^2 + y^2 \leq 1$.



Telo bomo parametrizirali s predpisom:

$$\begin{aligned} &\phi \in [0, 2\pi], \\ &r \in [0, 1], \\ &z \in [r^2, r], \end{aligned}$$

njegov volumen pa je

$$V = \int_0^{2\pi} d\phi \int_0^1 dr \int_{r^2}^{r} r\, dz = \ldots = \frac{\pi}{6}.$$

□

(14) Izračunaj integral funkcije $f(x, y, z) = \sqrt{x^2 + y^2}$ po delu valja $x^2 + y^2 \leq 9$, ki leži med ravninama $x + z = 5$ in $z = 1$.

Rešitev: Del valja $x^2 + y^2 \leq 9$, ki leži med ravninama $x + z = 5$ in $z = 1$, parametriziramo s predpisom:

$$\begin{aligned} &\phi \in [0, 2\pi], \\ &r \in [0, 3], \\ &z \in [1, 5 - r\cos\phi]. \end{aligned}$$

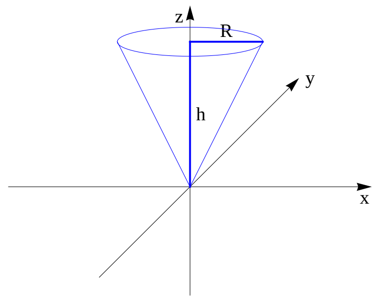
Ker je $f(x, y, z) = \sqrt{x^2 + y^2} = r$, dobimo:

$$\begin{aligned} \iiint_D \sqrt{x^2 + y^2}\, dx\, dy\, dz &= \int_0^{2\pi} d\phi \int_0^3 dr \int_{z=1}^{z=5-r\cos\phi} r\, r\, dz = \int_0^{2\pi} d\phi \int_0^3 dr \left(r^2 z\right)\Big|_1^{5-r\cos\phi}, \\ &= \int_0^{2\pi} d\phi \int_0^3 dr\, r^2(5 - r\cos\phi - 1), \\ &= \int_0^{2\pi} d\phi \int_0^3 (4r^2 - r^3\cos\phi)\, dr, \\ &= \int_0^{2\pi} d\phi \left(\frac{4r^3}{3} - \frac{r^4}{4}\cos\phi\right)\Bigg|_{r=0}^{r=3}, \\ &= \int_0^{2\pi} d\phi \left(36 - \frac{81}{4}\cos\phi\right) = \left(36\phi - \frac{81}{4}\sin\phi\right)\Big|_{\phi=0}^{\phi=2\pi}, \\ &= 72\pi. \end{aligned}$$

□

(15) Izračunaj središče stožca, omejenega s ploskvama $z = h$ in $z^2 = \frac{h^2}{R^2}(x^2 + y^2)$.

Rešitev: Računamo središče stožca z višino h in s polmerom osnovne ploskve R.



Koordinate središče telesa T so enake:

$$x_* = \frac{\iiint_T x\,dx\,dy\,dz}{V},$$
$$y_* = \frac{\iiint_T y\,dx\,dy\,dz}{V},$$
$$z_* = \frac{\iiint_T z\,dx\,dy\,dz}{V},$$

kjer je V volumen telesa T. Naš stožec bomo parametrizirali s cilindričnimi koordinatami:

$$\phi \in [0, 2\pi],$$
$$r \in [0, R],$$
$$z \in \left[\frac{h}{R}r, h\right].$$

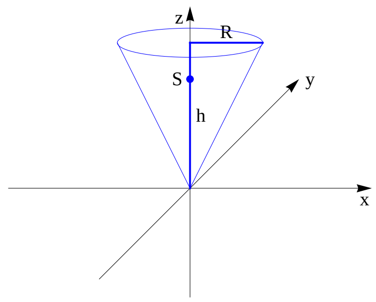
Zaradi rotacijske simetrije leži središče stožca na navpični osi, od koder sledi $x_* = y_* = 0$. Zato moramo izračunati samo z koordinato središča. Najprej je:

$$\begin{aligned}
\iiint_T z\,dx\,dy\,dz &= \int_0^{2\pi} d\phi \int_0^R dr \int_{\frac{h}{R}r}^{h} zr\,dz = \int_0^{2\pi} d\phi \int_0^R dr \left(r\frac{z^2}{2} \right) \Bigg|_{z=\frac{h}{R}r}^{z=h}, \\
&= \int_0^{2\pi} d\phi \int_0^R dr \left(\frac{rh^2}{2} - \frac{h^2r^3}{2R^2} \right) = \int_0^{2\pi} d\phi \left(\frac{r^2h^2}{4} - \frac{h^2r^4}{8R^2} \right) \Bigg|_{r=0}^{r=R}, \\
&= \int_0^{2\pi} d\phi \left(\frac{R^2h^2}{4} - \frac{h^2R^4}{8R^2} \right) = \left(\frac{R^2h^2}{4} - \frac{h^2R^2}{8} \right) \int_0^{2\pi} d\phi, \\
&= \frac{R^2h^2}{8} \cdot 2\pi = \frac{\pi R^2h^2}{4}.
\end{aligned}$$

Ker je volumen stožca enak $V = \frac{\pi R^2 h}{3}$, od tod sledi

$$z_* = \frac{\frac{\pi R^2h^2}{4}}{\frac{\pi R^2h}{3}} = \frac{3}{4}h.$$

Poglejmo še skico.


z
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