Matematika 2

Integrali s parametrom

(1) Izračunaj naslednje funkcije in njihove odvode:

(a) $F(x) = \int_0^x \cos t \, dt$,

(b) $F(x) = \int_0^{x^2} e^t \, dt$,

(c) $F(x) = \int_x^0 t^2 \, dt$.

Rešitev: (a) Računajmo

$$F(x) = \int_0^x \cos t \, dt = \sin t \Big|_0^x = \sin x - \sin 0 = \sin x.$$

Od tod sledi

$$F'(x) = \cos x.$$

Opazimo, da je odvod funkcije F enak funkciji, ki jo integriramo, da dobimo funkcijo F.
(b) Sedaj je

$$F(x) = \int_0^{x^2} e^t \, dt = e^t \Big|_0^{x^2} = e^{x^2} - e^0 = e^{x^2} - 1$$

in

$$F'(x) = 2xe^{x^2}.$$

(c) V tem primeru je

$$F(x) = \int_x^0 t^2 \, dt = \frac{t^3}{3} \Big|_x^0 = 0 - \frac{x^3}{3} = -\frac{x^3}{3}$$

in

$$F'(x) = -x^2.$$

Tokrat je odvod funkcije F enak funkciji, ki jo integriramo, le z negativnim predznakom.
□

(2) Naj bo $F(x) = \int_0^1 f(x,y) \, dy$, kjer je $f(x,y) = (\sqrt{x} + y)^2$.

(a) Izračunaj funkciji F in F'.

(b) Izračunaj funkcijo $G(x) = \int_0^1 \frac{\partial f}{\partial x}(x,y) \, dy$.

Rešitev: (a) Velja

$$f(x,y) = (\sqrt{x} + y)^2 = x + 2\sqrt{x}y + y^2.$$

Od tod dobimo

$$F(x) = \int_0^1 (x + 2\sqrt{x}y + y^2) \, dy = \left(xy + \sqrt{x}y^2 + \frac{y^3}{3} \right) \Big|_0^1 = \left(x + \sqrt{x} + \frac{1}{3} \right) - 0 = x + \sqrt{x} + \frac{1}{3}$$

in

$$F'(x) = 1 + \frac{1}{2\sqrt{x}}.$$

(b) Najprej izračunajmo parcialni odvod

$$\frac{\partial f}{\partial x}(x, y) = 1 + \frac{y}{\sqrt{x}}.$$

Od tod dobimo

$$G(x) = \int_0^1 \left(1 + \frac{y}{\sqrt{x}}\right) dy = \left(y + \frac{y^2}{2\sqrt{x}}\right)\Big|_0^1 = \left(1 + \frac{1}{2\sqrt{x}}\right) - 0 = 1 + \frac{1}{2\sqrt{x}}.$$

Vidimo, da lahko odvod funkcije F izračunamo tako, da integriramo parcialni odvod funkcije f. □

(3) Izračunaj odvode naslednjih funkcij:

(a) $F(x) = \displaystyle\int_1^3 \frac{e^{tx}}{t}\, dt$,

(b) $F(x) = \displaystyle\int_{2x}^{x^2} \sin(2u + 3)\, du$,

(c) $G(y) = \displaystyle\int_{3y}^{7} (x + y)^2\, dx$.

Rešitev: V splošnem lahko integral s parametrom odvajamo s pomočjo naslednje formule. Če je funkcija F definirana s predpisom

$$F(x) = \int_{\alpha(x)}^{\beta(x)} f(x, y)\, dy,$$

je njen odvod enak

$$F'(x) = \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x}(x, y)\, dy + \beta'(x) f(x, \beta(x)) - \alpha'(x) f(x, \alpha(x)).$$

(a) Imamo $\alpha(x) = 1$, $\beta(x) = 3$ in $f(x, t) = \frac{e^{tx}}{t}$. Torej je $\alpha'(x) = \beta'(x) = 0$, $\frac{\partial f}{\partial x}(x, t) = e^{tx}$ in

$$F'(x) = \int_1^3 e^{tx}\, dt + 0 - 0 = \frac{1}{x} e^{tx}\Big|_1^3 = \frac{1}{x}\left(e^{3x} - e^x\right).$$

(b) Sedaj je $\alpha(x) = 2x$, $\beta(x) = x^2$ in $f(x, u) = \sin(2u + 3)$. Sledi $\alpha'(x) = 2$, $\beta'(x) = 2x$, $\frac{\partial f}{\partial x}(x, u) = 0$ in

$$F'(x) = 0 + 2x f(x, x^2) - 2 f(x, 2x) = 2x \sin(2x^2 + 3) - 2\sin(4x + 3).$$

(c) Tokrat imamo $\alpha(y) = 3y$, $\beta(y) = 7$ in $g(x, y) = x^2 + 2xy + y^2$, kar nam da $\alpha'(y) = 3$, $\beta'(y) = 0$, $\frac{\partial g}{\partial y}(x, y) = 2x + 2y$ in

$$\begin{aligned} G'(y) &= \int_{3y}^{7} (2x + 2y)\, dx + 0 - 3g(3y, y) = (x^2 + 2xy)\Big|_{3y}^{7} - 3(3y + y)^2, \\ &= (49 + 14y) - ((3y)^2 + 6y^2) - 3(4y)^2 = 49 + 14y - 15y^2 - 48y^2, \\ &= -63y^2 + 14y + 49. \end{aligned}$$

□

(4) Izračunaj integral funkcije $F(y) = \int\limits_1^2 \frac{dx}{(x+y)^2}$ v mejah od 3 do 4.

Rešitev: Najprej bomo izračunali predpis za funkcijo F, nato pa še integral.

$$F(y) = \int\limits_1^2 \frac{dx}{(x+y)^2} = -\frac{1}{x+y}\Big|_1^2 = -\frac{1}{2+y} + \frac{1}{1+y}.$$

Od tod sledi:

$$\int_3^4 F(y)\,dy = \int_3^4 \left(-\frac{1}{2+y} + \frac{1}{1+y}\right) dy = (-\ln(2+y) + \ln(1+y))\Big|_3^4,$$
$$= (-\ln 6 + \ln 5) - (-\ln 5 + \ln 4) = \ln \frac{25}{24}.$$

□

(5) Izrazi naslednje integrale s pomočjo funkcije gama:

(a) $\int_0^\infty t^3 e^{-t}\,dt,$

(b) $\int_0^\infty t^{\frac{5}{2}} e^{-t}\,dt,$

(c) $\int_0^\infty x^{2n} e^{-x^2}\,dx,$

(d) $\int_0^\infty e^{-x^n}\,dx,$

(e) $\int_{-\infty}^\infty e^{-x^2}\,dx.$

Rešitev: Funkcija gama je definirana s predpisom

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\,dx$$

za vse $s > 0$. Velja $\Gamma(1) = 1$ in $\Gamma(1/2) = \sqrt{\pi}$, nekatere druge vrednosti funkcije Γ pa lahko izračunamo z rekurzivnim predpisom $\Gamma(s+1) = s\Gamma(s)$.

(a) $\int_0^\infty t^3 e^{-t}\,dt$:

Imamo integral, ki sovpada z definicijo funkcije gama.

$$\int_0^\infty t^3 e^{-t}\,dt = \int_0^\infty t^{4-1} e^{-t}\,dt = \Gamma(4) = 3\Gamma(3) = 3 \cdot 2\Gamma(2) = 6 \cdot 1\Gamma(1) = 6.$$

(b) $\int_0^\infty t^{\frac{5}{2}} e^{-t}\, dt$:

Tudi tokrat si lahko pomagamo z definicijo funkcije gama.

$$\int_0^\infty t^{\frac{5}{2}} e^{-t}\, dt = \int_0^\infty t^{\frac{7}{2}-1} e^{-t}\, dt = \Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{15}{4}\Gamma\left(\frac{3}{2}\right) = \frac{15}{8}\Gamma\left(\frac{1}{2}\right) = \frac{15\sqrt{\pi}}{8}.$$

(c) $\int_0^\infty x^{2n} e^{-x^2}\, dx$:

V tem primeru bomo najprej z uvedbo nove spremenljivke integral prevedli v ustrezno obliko. Definirajmo $t = x^2$. Potem je $dt = 2x\, dx$. Od tod lahko izrazimo $dx = \frac{dt}{2\sqrt{t}}$. Ko je $x = 0$, je $t = 0$, ko gre $x \to \infty$ pa gre prav tako tudi $t \to \infty$. Sledi

$$\int_0^\infty x^{2n} e^{-x^2}\, dx = \int_0^\infty t^n e^{-t} \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty t^{n-\frac{1}{2}} e^{-t}\, dt.$$

Sedaj lahko uporabimo definicijo funkcije gama.

$$\int_0^\infty x^{2n} e^{-x^2}\, dx = \frac{1}{2}\int_0^\infty t^{n-\frac{1}{2}} e^{-t}\, dt = \frac{1}{2}\int_0^\infty t^{\left(n+\frac{1}{2}\right)-1} e^{-t}\, dt = \frac{1}{2}\Gamma\left(n+\frac{1}{2}\right).$$

(d) $\int_0^\infty e^{-x^n}\, dx$:

Sedaj definirajmo $t = x^n$. Potem je $dt = nx^{n-1}\, dx$ in $dx = \frac{dt}{n\sqrt[n]{t^{n-1}}}$. Ko je $x = 0$, je $t = 0$, ko gre $x \to \infty$ pa gre prav tako tudi $t \to \infty$. Sledi

$$\int_0^\infty e^{-x^n}\, dx = \int_0^\infty e^{-t} \frac{dt}{n\sqrt[n]{t^{n-1}}} = \frac{1}{n}\int_0^\infty t^{-\frac{n-1}{n}} e^{-t}\, dt = \frac{1}{n}\int_0^\infty t^{\frac{1}{n}-1} e^{-t}\, dt = \frac{1}{n}\Gamma\left(\frac{1}{n}\right).$$

(e) $\int_{-\infty}^\infty e^{-x^2}\, dx$:

Sedaj ne integriramo po intervalu $[0, \infty)$ ampak po intervalu $(-\infty, \infty)$. Ker pa je funkcija $f(x) = e^{-x^2}$ soda, je

$$\int_{-\infty}^\infty e^{-x^2}\, dx = 2\int_0^\infty e^{-x^2}\, dx.$$

Ta integral smo že izračunali pri prejšnji nalogi. Sledi

$$\int_{-\infty}^\infty e^{-x^2}\, dx = 2\int_0^\infty e^{-x^2}\, dx = 2 \cdot \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

□

(6) Izrazi naslednja integrala s pomočjo funkcije beta:

(a) $\int_0^1 t^{-\frac{2}{3}}(1-t)^{-\frac{1}{2}}\, dt,$

(b) $\int_0^1 t^2\sqrt{1-t^2}\, dt.$

Dokaz. Funkcijo beta definiramo z integralom

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx$$

za vse pozitivne p in q. Funkcija B je simetrična, izrazimo pa jo lahko tudi s pomočjo funkcije gama v obliki

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Za $0<p<1$ lahko eksplicitno izračunamo, da velja

$$B(p,1-p)=\frac{\pi}{\sin p\pi}.$$

(a) $\int_0^1 t^{-\frac{2}{3}}(1-t)^{-\frac{1}{2}}\,dt$:

Ta integral lahko izračunamo direktno po definiciji.

$$\int_0^1 t^{-\frac{2}{3}}(1-t)^{-\frac{1}{2}}\,dt=\int_0^1 t^{\frac{1}{3}-1}(1-t)^{\frac{1}{2}-1}\,dt=B\left(\frac{1}{3},\frac{1}{2}\right).$$

(b) $\int_0^1 t^2\sqrt{1-t^2}\,dt$:

Uvedimo novo spremenljivko $x=t^2$, kar nam da $dx=2t\,dt$ in $dt=\frac{dx}{2\sqrt{x}}$. Pri $t=0$ je $x=0$, pri $t=1$ pa je $x=1$. Sledi

$$\begin{aligned}\int_0^1 t^2\sqrt{1-t^2}\,dt&=\int_0^1 x\sqrt{1-x}\,\frac{dx}{2\sqrt{x}}=\frac{1}{2}\int_0^1\sqrt{x}\sqrt{1-x}\,dx=\frac{1}{2}\int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}\,dx,\\&=\frac{1}{2}B\left(\frac{3}{2},\frac{3}{2}\right)=\frac{1}{2}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}=\frac{1}{2}\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)}=\frac{\pi}{16}.\end{aligned}$$

□

(7) Izrazi naslednje integrale s pomočjo funkcije beta:

(a) $\int_0^{\frac{\pi}{2}}\sin^5 x\,dx,$

(b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^5 x\,dx,$

(c) $\int_0^{\frac{\pi}{2}}\sin^3 x\cos^2 x\,dx,$

(d) $\int_0^{\frac{\pi}{2}}\operatorname{tg}^{\frac{1}{3}} x\,dx.$

Rešitev: Funkcija beta nam omogoča izračunati tudi nekatere določene integrale trigonometričnih funkcij. Pri tem uporabljamo formulo

$$\int_0^{\frac{\pi}{2}} \sin^p x \cos^q x \, dx = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right).$$

(a) $\int_0^{\frac{\pi}{2}} \sin^5 x \, dx$:

Računajmo

$$\int_0^{\frac{\pi}{2}} \sin^5 x \, dx = \frac{1}{2} B\left(3, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} = \frac{1}{2} \cdot \frac{2 \cdot 1 \cdot \Gamma(1) \cdot \Gamma\left(\frac{1}{2}\right)}{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)} = \frac{8}{15}.$$

(b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, dx$:

Integracijski interval tokrat ni enak kot v naši formuli. Z upoštevanjem dejstva, da je cos soda funkcija, pa dobimo

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5 x \, dx = 2 \int_0^{\frac{\pi}{2}} \cos^5 x \, dx = 2 \cdot \frac{1}{2} B\left(\frac{1}{2}, 3\right) = B\left(3, \frac{1}{2}\right) = \frac{16}{15}.$$

(c) $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, dx$:

Sedaj imamo

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x \, dx = \frac{1}{2} B\left(2, \frac{3}{2}\right) = \frac{1}{2} \frac{\Gamma(2)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} = \frac{1}{2} \cdot \frac{1 \cdot \Gamma(1) \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)}{\frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)} = \frac{2}{15}.$$

(d) $\int_0^{\frac{\pi}{2}} \operatorname{tg}^{\frac{1}{3}} x \, dx$:

Pri tem integralu bomo upoštevali, da velja $\operatorname{tg} x = \frac{\sin x}{\cos x}$.

$$\int_0^{\frac{\pi}{2}} \operatorname{tg}^{\frac{1}{3}} x \, dx = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{3}} x \cos^{-\frac{1}{3}} x \, dx = \frac{1}{2} B\left(\frac{2}{3}, \frac{1}{3}\right) = \frac{1}{2} \cdot \frac{\pi}{\sin \frac{2\pi}{3}} = \frac{1}{2} \cdot \frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{\pi\sqrt{3}}{3}.$$

□