DOMAČA NALOGA:
VRSTE

1. Dano je zaporedje s splošnim členom $a_k = \frac{1}{k(k+1)}$. Izračunaj vsoto vrste $\sum_{k=0}^{\infty} a_k$.

   Rešitev: Vsota vrste je enaka 1.

2. Izračunaj vsoto vrste
$$1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots =$$

   Rešitev: $\frac{5}{4}$

3. Izračunaj vsoto vrste
$$\sum_{n=0}^{\infty} \frac{3^n + 2^n}{5^n}.$$

   Rešitev: $\frac{25}{6}$

4. Reši enačbo
$$x + x^3 + x^5 + x^7 + \cdots = \frac{3}{8}.$$

   Rešitev: $x = \frac{1}{3}$

5. S pomočjo primerjalnega kriterija preveri ali naslednji vrsti konvergirata

   (a) $\sum_{n=0}^{\infty} \frac{1}{n2^n}$

   (b) $\sum_{n=0}^{\infty} \frac{1}{2n-1}$

   Rešitev: V primeru (a) vrsta konvergira in v (b) divergira.

6. S pomočjo kvocientnega ali korenskega kriterija preveri, ali naslednji vrsti konvergirata

   (a) $\sum_{n=0}^{\infty} \frac{2n-1}{2^n}$

   (b) $\sum_{n=0}^{\infty} \frac{\sin^n 20}{(2+\frac{1}{n})^n}$

Rešitev: V obeh primerih je vrsta konvergentna.

7. Ali vrsta $\sum_{n=0}^{\infty}(-1)^{\frac{n(n-1)}{2}}\left(\frac{n}{2n-1}\right)^n$ konvergira in ali konvergira absolutno?

   Rešitev: Vrsta konvergira in absolutno konvergira.

8. Ali vrsta $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ absolutno konvergira za poljubno realno število x?

   Rešitev: Da.

9. Z uporabo integralskega kriterija dokaži, da vrsta $\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n+1}}$ divergira.

10. Funkcijo $f(x)=x^4-7x^3+19x^2-23x+11$ razvij v Taylorjevo vrsto okoli točke $a=2$.

    Rešitev: $f(x)=1+(x-2)+(x-2)^2+(x-2)^3+(x-2)^4$

11. Funkcijo $f(x)=\frac{1}{\sqrt{x}}$ razvij v Taylorjevo vrsto okoli točke $a=1$.

    Rešitev: $f(x)=1-\frac{x-1}{2\cdot 1!}+\frac{(x-1)^2}{4\cdot 2!}-\frac{(x-1)^3}{8\cdot 3!}+\cdots=\sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^n}{2^n\cdot n!}$

12. Funkcijo $f(x)=\ln(1+x)$ razvij v Taylorjevo vrsto okoli točke $a=0$ in izračunaj vsoto vrste $\sum_{n=1}^{\infty}(-1)^n\frac{1}{2^n n}$.

    Rešitev: $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$ in $\sum_{n=1}^{\infty}(-1)^n\frac{1}{2^n n}=\ln\frac{2}{3}$

(2)

(2) $1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\ldots = ?$

↳ geometrijska vrsta

$a=1$

$q=\frac{1}{4}$ $|q|<1 \Rightarrow \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n = \frac{a}{1-q} = \frac{1}{1-\frac{1}{4}} =$

$= \frac{1}{\frac{3}{4}} = \underline{\underline{\frac{4}{3}}}$

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(3) $\sum_{n=0}^{\infty} \frac{3^n+2^n}{5^n}$

2 geom. vrsti!

$q_1=\frac{3}{5}$ $a_1=1$

$q_2=\frac{2}{5}$ $a_2=1$

$\sum_{n=0}^{\infty} \frac{3^n+2^n}{5^n} = \sum_{n=0}^{\infty}\left(\frac{3}{5}\right)^n + \sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^5$

$= \frac{1}{1-\frac{3}{5}} + \frac{1}{1-\frac{2}{5}} = \frac{1}{\frac{2}{5}} + \frac{1}{\frac{3}{5}} = \frac{5}{2} + \frac{5}{3} = \underline{\underline{\frac{25}{6}}}$

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(4) $x + x^3 + x^5 + x^7 + \ldots = \frac{3}{8}$

geometrijska vrsta $\boxed{\begin{matrix} a=x \\ q=x^2 \end{matrix}}$

$\frac{a}{1-q} = \frac{3}{8}$

mora biti med $(-1,1)$

$q=x^2$

$\frac{x}{1-x^2} = \frac{3}{8}$

$\boxed{\cancel{x_1 = -\frac{18}{6} = -3}}$

$3-3x^2=8x$

$3x^2+8x-3=0$

$\boxed{x_2 = \frac{2}{6} = \frac{1}{3}}$

$x_{1,2} = \frac{-8 \pm \sqrt{100}}{6} = \frac{-8 \pm 10}{6}$

$64+4\cdot 9=100$

↳ prava rešitev

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1. $a_k = \frac{1}{n(n+1)}$, $\sum_{k=0}^{\infty} \frac{1}{n(n+1)} = ?$

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

$S_3 = \frac{2}{3} + \frac{1}{3 \cdot 4} = \frac{2}{3} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$ $\boxed{S_n = \frac{n}{n+1}}$

$S_4 = \frac{3}{4} + \frac{1}{4 \cdot 5} = \frac{3}{4} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5}$

Indukcija : 1) $n=1 \Rightarrow LS = S_1 = \frac{1}{2}$, $DS = \frac{1}{1+1} = \frac{1}{2}$ $LS = DS \Rightarrow$ velja za $n=$

2) $n \overset{?}{\Rightarrow} n+1$

$S_{n+1} = \frac{n+1}{n+2}$

$LS = S_n + a_{n+1} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} =$

$= \frac{(n+1)^{\cancel{2}}}{\cancel{(n+1)}(n+2)} = \frac{n+1}{n+2} = DS \Rightarrow$ velja za vsak $n \in \mathbb{N}$

$\Rightarrow S_n = \frac{n}{n+1}$ je vsota!

$\sum_{n=0}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n+1-1}{n+1} = \lim_{n\to\infty} \left(1 - \cancel{\frac{1}{n+1}}^{0}\right)$

$\sum_{n=0}^{\infty} \frac{1}{n(n+1)} = \underline{\underline{1}}$

(5) a) $\sum_{n=0}^{\infty} \frac{1}{n2^n} = a_n$

$\frac{1}{n2^n} > 0 \quad \frac{1}{n2^n} < \frac{1}{2^n}$

$0 < \frac{1}{n2^n} < \frac{1}{2^n}$

$\frac{1}{2^n} = \left(\frac{1}{2}\right)^n \rightarrow$ geometrijska vrsta

$\sum \left(\frac{1}{2}\right)^n$ - konvergira $\Rightarrow$ $\boxed{\sum_{n=0}^{\infty} \frac{1}{n2^n} \text{ - konvergira tudi}}$

b) $\sum_{n=0}^{\infty} \frac{1}{2n-1}$

$0 < \frac{1}{2n-1} < \frac{1}{n} \quad \sum_{n=1}^{\infty} \frac{1}{n}$ divergira $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{2n-1}$ - diverg. tudi

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(6) a) $\sum_{n=0}^{\infty} \frac{2n-1}{2^n}$

$$L = \lim_{n\to\infty} \frac{\frac{2(n+1)-1}{2^{n+1}}}{\frac{2n-1}{2^n}} = \lim_{n\to\infty} \frac{\cancel{2^n}(2n+1)}{2\cdot\cancel{2^n}(2n-1)} = \lim_{n\to\infty} \frac{2n+1}{4n-2}$$

$$L = \lim_{n\to\infty} \frac{2 \cancel{+\frac{1}{n}}}{4 - \cancel{\frac{2}{n}}} = \frac{1}{2} \quad L < 1 \Rightarrow \sum_{n=0}^{\infty} \frac{2n-1}{2^n} \text{ KONVERG.}$$

b) $\sum_{n=0}^{\infty} \frac{\sin^n 20}{(2+\frac{1}{n})^n}$

$$L = \lim_{n\to\infty} \sqrt[n]{\left(\frac{\sin 20}{2+\frac{1}{n}}\right)^n} = \lim_{n\to\infty} \frac{\overset{<1}{\sin 20}}{2+\cancel{\frac{1}{n}}^{\,0}} < \frac{1}{2} < 1 \Rightarrow L < 1$$

$$\Rightarrow \sum \frac{\sin^n 20}{(2+\frac{1}{n})^n} \text{ KONVERG.}$$

(9) $\sum_{u=1}^{\infty} \frac{1}{\sqrt{2u+1}}$ - divergira

$$\int_1^{\infty} \frac{1}{\sqrt{2x+1}}\,dx = \lim_{b\to\infty} \int_1^{b} (2x+1)^{-\frac{1}{2}}\,dx = \lim_{b\to\infty} \frac{1}{2}\cdot 2\cdot (2x+1)^{\frac{1}{2}}\Big|_1^{b} =$$

$$= \lim_{b\to\infty} \sqrt{2x+1}\Big|_1^{b} = \lim_{b\to\infty}\left(\sqrt{2b+1} - \sqrt{3}\right) = \infty$$

Ker je $\int_1^{\infty} \frac{1}{\sqrt{2n+1}}\,dn = \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n+1}}$ ne obstaja $\Rightarrow$ **DIVERGIRA**

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(10) $f(x) = x^4 - 7x^3 + 19x^2 - 23x + 11$ $\qquad a=2$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$f(a) = f(2) = 16 - 7\cdot 8 + 19\cdot 4 - 23\cdot 2 + 11 = \boxed{1}$

$f'(x) = 4x^3 - 21x^2 + 38x - 23$ $\qquad f'(2) = 4\cdot 8 - 21\cdot 4 + 38\cdot 2 - 23 = \boxed{1}$

$f''(x) = 12x^2 - 42x + 38$ $\qquad f''(2) = 12\cdot 4 - 42\cdot 2 + 38 = \boxed{2}$

$f'''(x) = 24x - 42$ $\qquad f'''(2) = 24\cdot 2 - 42 = \boxed{6}$

$f^{IV}(x) = 24$ $\qquad f^{IV}(2) = 24$

$$f(x) = 1 + \frac{1}{1!}(x-2) + \frac{\cancel{2}}{\cancel{2!}}(x-2)^2 + \frac{\cancel{6}}{\cancel{3!}}(x-2)^3 + \frac{\cancel{24}}{\cancel{4!}}(x-3)^4$$

$$\boxed{f(x) = 1 + (x-2) + (x-2)^2 + (x-2)^3 + (x-2)^4}$$

(11) $f(x) = \frac{1}{\sqrt{x}} \quad a=1 \qquad f(1) = 1$

$f(x) = x^{-\frac{1}{2}}$

$f'_{(x)} = -\frac{1}{2} \cdot x^{-\frac{3}{2}} \qquad f'(1) = -\frac{1}{2}$

$f''_{(x)} = \frac{3}{4} x^{-\frac{5}{2}} \qquad f''(1) = \frac{3}{4}$

$f'''_{(x)} = -\frac{15}{8} x^{-\frac{7}{2}} \qquad f'''(1) = -\frac{15}{8}$

$$\frac{1}{\sqrt{x}} = 1 - \frac{1}{2} \cdot (x-1) + \frac{3}{4} \cdot \frac{(x-1)^2}{2!} - \frac{15}{8} \frac{(x-1)^3}{3!} + \dots$$

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(12) $f(x) = \ln(1+x) \qquad a=0 \qquad \sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n n} = ?$

$f(0) = 0 \qquad \sum_{n=1}^{\infty} (-1) \frac{1}{2^n n} = ? \quad n=1$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1} \quad f'(0) = 1 \qquad x = -\frac{1}{2}$

$f''(x) = -(1+x)^{-2} \qquad f''(0) = -1 \qquad \ln(1-\frac{1}{2}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (-1)^n}{2^n \cdot n}$

$f'''(x) = 2(1+x)^{-3} \qquad f'''(0) = 2 \qquad \ln(\frac{1}{2}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n}$

$f^{IV}(x) = -6(1+x)^{-4} \qquad f^{IV}(0) = -6$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n n} = \ln \frac{1}{2}$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \dots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}$$